Sentential Logic

Excerpted From
Symbolic Logic: An Accessible Introduction to Serious Mathematical Logic

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Preface

There is, I think, a gap between what many students learn in their first course in formal logic, and what they are expected to know for their second. While courses in mathematical logic with metalogical components often cast only the barest glance at mathematical induction or even the very idea of reasoning from definitions, a first course may also leave these untreated, and fail explicitly to lay down the definitions upon which the second course is based. The aim of this text is to integrate material from these courses and, in particular, to make serious mathematical logic accessible to students I teach. The first parts introduce classical symbolic logic as appropriate for beginning students; the last parts build to Gödel's adequacy and incompleteness results. A distinctive feature of the last section is a complete development of Gödel's second incompleteness theorem.

Accessibility, in this case, includes components which serve to locate this text among others: First, assumptions about background knowledge are minimal. I do not assume particular content about computer science, or about mathematics much beyond high school algebra. Officially, everything is introduced from the ground up. No doubt, the material requires a certain sophistication — which one might acquire from other courses in critical reasoning, mathematics or computer science. But the requirement does not extend to particular contents from any of these areas.

Second, I aim to build skills, and to keep conceptual distance for different applications of 'so' relatively short. Authors of books that are completely correct and precise may assume skills and require readers to recognize connections not fully explicit. It may be that this accounts for some of the reputed difficulty of the material. The results are often elegant. But this can exclude a class of students capable of grasping and benefiting from the material, if only it is adequately explained. Thus I attempt explanations and examples to put the student at every stage in a position to understand the next. In some cases, I attempt this by introducing relatively concrete methods for reasoning. The methods are, no doubt, tedious or unnecessary for the experienced logician. However, I have found that they are valued by students, insofar as students are presented with an occasion for success. These methods are not meant to wash over or substitute for understanding details, but rather to expose and clarify them. Clarity, beauty and power come, I think, by getting at details, rather than burying or ignoring them.

Third, the discussion is ruthlessly directed at core results. Results may be rendered inaccessible to students, who have many constraints on their time and schedules, simply because the results would come up in, say, a second course rather than a first. My idea is to exclude side topics and problems, and to go directly after (what I see as) the core. One manifestation is the way definitions and results from earlier sections feed into ones that follow. Thus simple integration is a benefit. Another is the way predicate logic with identity is introduced as a whole in Part I. Though it is possible to isolate sentential logic from the first parts of chapter 2 through chapter 7, and so to use the text for separate treatments of sentential and predicate logic, the guiding idea is to avoid repetition that would be associated with independent treatments for sentential logic, or perhaps monadic predicate logic, the full predicate logic, and predicate logic with identity.

Also (though it may suggest I am not so ruthless about extraneous material as I would like to think), I try to offer some perspective about what is accomplished along the way. In addition, this text may be of particular interest to those who have, or desire, an exposure to natural deduction in formal logic. In this case, accessibility arises from the nature of the system, and association with what has come before. In the first part, I introduce both axiomatic and natural derivation systems; and in **??**, show how they are related.

There are different ways to organize a course around this text. For students who are likely to complete the whole, the ideal is to proceed sequentially through the text from beginning to end (but postponing chapter 3 until after chapter 6). Taken as wholes, Part II depends on Part I; Parts **??** and **??** on Parts I and II. **??** is mostly independent of **??**. I am currently working within a sequence that isolates sentential logic from quantificational logic, treating them in separate quarters, together covering all of chapters 1 - 7 (except 3). A third course picks up leftover chapters from the first two parts (3 and **??**) with **??**; and a fourth the leftover chapters from the first parts with **??**. Perhaps not the most efficient arrangement, but the best I have been able to do with shifting student populations. Other organizations are possible!

A remark about chapter 7 especially for the instructor: By a formal system for reasoning with semantic definitions, chapter 7 aims to leverage derivation skills from earlier chapters to informal reasoning with definitions. I have had a difficult time convincing instructors to try this material — and even been told flatly that these skills "cannot be taught." In my experience, this is false (and when I have been able to convince others to try the chapter, they have quickly seen its value). Perhaps the difficulty is just that the strategy is unfamiliar. Of course, if one is presented with students whose mathematical sophistication is sufficient for advanced work, it is not necessary. But if, as is often the case especially for students in philosophy, one obtains one's mathematical sophistication *from* courses in logic, this chapter is an important part of the bridge from earlier material to later. Additionally, the chapter is an important "take-away" even for students who will not continue to later material. The chapter closes an open question from chapter 4 — how it is possible to demonstrate quantificational validity. But further, the ability to reason closely with definitions is a skill from which students in (sentential or) predicate logic, even though they never go on to formalize another sentence or do another derivation, will benefit both in philosophy and more generally.

Another remark about the (long) sections **??**, **??** and **??**. These develop in PA the "derivability conditions" for Gödel's second theorem. They are perhaps for enthusiasts. Still, in my experience many students are enthusiasts and, especially from an introduction, benefit by seeing how the conditions are derived. There are different ways to treat the sections. One might work through them in some detail. However, even if you decide to pass them by, there is an advantage having a panorama at which to wave and say "thus it is accomplished!"

Naturally, results in this book are not innovative. If there is anything original, it is in presentation. Even here, I am greatly indebted to others, especially perhaps Bergmann, Moor and Nelson, *The Logic Book*, Mendelson, *Introduction to Mathematical Logic*, and Smith, *An Introduction to Gödel's Theorems*. I thank my first logic teacher, G.J. Mattey, who communicated to me his love for the material. And I thank especially my colleagues John Mumma and Darcy Otto for many helpful comments. Hannah Baehr and Catlin Andrade made comments and produced answers to exercises for certain parts. In addition I have received helpful feedback from Steve Johnson, along with students in different logic classes at CSUSB. I welcome comments, and expect that your sufferings will make it better still.

This text evolved over a number of years starting modestly from notes originally provided as a supplement to other texts. It is now long (!) and perhaps best conceived in separate volumes for Parts I and II and then Parts **??** and **??**. With the addition of **??** it now complete. (But **??**, which I rarely get to in teaching, remains a stub that could be developed in different directions.) Most of the text is reasonably stable, though I shall be surprised if I have not introduced errors in the last part both substantive and otherwise.

I think this is fascinating material, and consider it great reward when students respond "cool!" as they sometimes do. I hope you will have that response more than once along the way.

T.R.
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Contents

Named Definitions

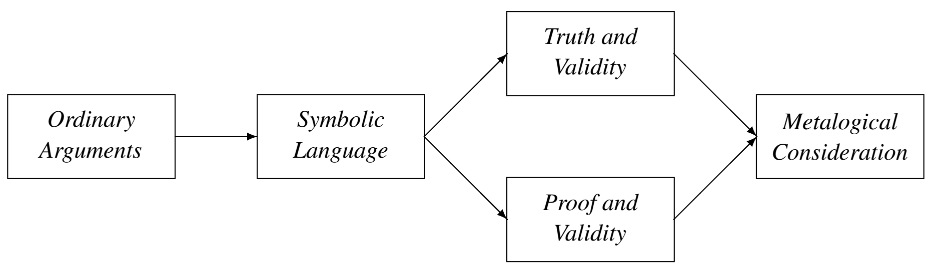
Quick Reference Guides

Part I

The Elements: Four Notions of Validity

Introductory

Symbolic logic is a tool for argument evaluation. In this part of the text we introduce the basic elements of that tool. Those parts are represented in the following diagram.



The starting point is ordinary arguments. Such arguments come in various forms and contexts — from politics and ordinary living, to mathematics and philosophy. Here is a classic, simple case.

(A)
All men are mortal.
Socrates is a man.
———
Socrates is mortal.

This argument has *premises* listed above a line, with a *conclusion* listed below. The premises are supposed to demonstrate the conclusion. Here is another case which may seem less simple.

(B) If the maid did it, then it was done with a revolver only if it was done in the parlor. But if the butler is innocent, then the maid did it unless it was done in the parlor. The maid did it only if it was done with a revolver, while the butler is guilty if it did happen in the parlor. So the butler is guilty.

It is fun to think about this; from the given evidence, it follows that the butler did it! Here is an argument that is both controversial and significant.

(C) There is evil. If god is good, then there is no evil unless god has morally sufficient reasons for allowing it. If god is both omnipotent and omniscient, then god does not have morally sufficient reasons for allowing evil. So god is not good, omnipotent and omniscient.

A being is *omnipotent* if it is all-powerful, and *omniscient* if all-knowing. This is a version of the famous "problem of evil" for traditional theism. It matters whether the conclusion is true! Roughly, an argument is good if it does what it is supposed to do, if its premises demonstrate the conclusion, and bad if they do not. So a theist (someone who believes in god) may hold that (C) is a bad argument, but an atheist (someone who does not believe in god) that it is good.

We begin in chapter 1 with an account of success for ordinary arguments (the leftmost box). So we say what it is for an argument to be good or bad. This introduces us to the fundamental notions of *logical validity* and *logical soundness*. These will be our core concepts for argument evaluation. But just as it is one thing to know what integrity is, and another to know whether someone has it, so it is one thing to know what logical validity and logical soundness are, and another to know whether arguments have them. In some cases, it may be obvious. But others are not so clear — as, for example, cases (B) or (C) above, along with complex arguments in mathematics. Thus symbolic logic is introduced as a sort of machine or tool to identify validity and soundness.

This machine begins with certain symbolic representations of ordinary arguments (the box second from the left). That is why it is *symbolic* logic. We introduce these representations in chapter 2, and translate from ordinary arguments to the symbolic representations in chapter 5. Once arguments have this symbolic representation, there are different methods of operating upon them.

An account of truth and validity is developed for the symbolic representations in chapter 4 and chapter 7 (the upper box). On this account, truth and validity are associated with clearly defined criteria for their evaluation. And validity from this upper box implies logical validity for the ordinary arguments that are symbolically represented. Thus we obtain clearly defined criteria to identify the logical validity of arguments we care about. Evaluation of validity for the butler and evil cases is entirely routine given the methods from chapter 2, chapter 4 and chapter 5 — though the soundness of (C) will remain controversial!

Accounts for proof and validity are developed for the symbolic representations in chapter 3 and chapter 6 (the lower box). Again, on this account, proof and validity are associated with clearly defined criteria for their evaluation. And validity from the lower box implies logical validity for the ordinary arguments that are symbolically represented. The result is another well-defined approach to the identification of logical validity. Evaluation of validity for the butler and evil cases is entirely routine given the methods from, say, chapter 2, chapter 3 and chapter 5, or alternatively, chapter 2, chapter 5 and chapter 6 — though, again, the soundness of (C) will remain controversial.

These, then, are the elements of our logical "machine" — we start with the fundamental notion of logical validity; then there are symbolic representations of ordinary reasonings, along with approaches to evaluation from truth and validity, and from proof and validity. These elements are developed in this part. In later parts we turn to thinking about how these parts work together (the right-hand box). In particular, we begin thinking *how* to reason about logic (Part II), *demonstrate* that the same arguments come out valid by the truth method and by the proof method (**??**), and establish limits on application of logic and computing to arithmetic (**??**). But first we have to say what the elements are. And that is the task we set ourselves in this part.

Chapter 1

Logical Validity and Soundness

Symbolic logic is a tool or machine for the identification of argument goodness. It makes sense to begin, however, not with the machine, but by saying something about this argument goodness that the machinery is supposed to identify. That is the task of this chapter.

But first, we need to say what an argument is. An argument is made up of sentences one of which is taken to be supported by the others.

AR An *argument* is some sentences, one of which (the *conclusion*) is taken to be supported by the remaining sentences (the *premises*).

(Important definitions are often offset and given a short name as above; then there may be appeal to the definition by its name, in this case, 'AR'.) So an argument has premises which are taken to support a conclusion. Such support is often indicated by words or phrases of the sort, 'so', 'it follows', 'therefore', or the like. We will typically indicate the division by a simple line between premises and conclusion. Roughly, an argument is good if the premises do what they are taken to do, if they actually support the conclusion. An argument is bad if they do not accomplish what they are taken to do, if they do not actually support the conclusion.

Logical validity and soundness correspond to different ways an argument can go wrong. Consider the following two arguments:

(A)
Only citizens can vote
Hannah is a citizen
―――――
Hannah can vote

(B)
All citizens can vote
Hannah is a citizen
―――――
Hannah can vote

The line divides premises from conclusion, indicating that the premises are supposed to support the conclusion. Thus these are arguments. But these arguments go wrong in different ways. The premises of argument (A) are true; as a matter of fact, only citizens can vote, and Hannah (my daughter) is a citizen. But she cannot vote; she is not old enough. So the conclusion is false. Thus, in argument (A), the relation between the premises and the conclusion is defective. Even though the premises are true, there is no guarantee that the conclusion is true as well. We will say that this argument is *logically invalid*. In contrast, argument (B) is logically valid. If its premises were true, the conclusion would be true as well. So the *relation* between the premises and conclusion is not defective. The problem with this argument is that the premises are not true — not all citizens can vote. So argument (B) is defective, but in a different way. We will say that it is *logically unsound*.

The task of this chapter is to define and explain these notions of logical validity and soundness. I begin with some preliminary notions, then turn to official definitions of logical validity and soundness, and finally to some consequences of the definitions.

1.1 Consistent Stories

Given a certain notion of a *possible* or *consistent* story, it is easy to state definitions for logical validity and soundness. So I begin by identifying the kind of stories that matter. Then we will be in a position to state the definitions, and apply them in some simple cases.

Let us begin with the observation that there are different sorts of possibility. Consider, say, "Hannah could make it in the WNBA." This seems true. She is reasonably athletic, and if she were to devote herself to basketball over the next few years, she might very well make it in the WNBA. But wait! Hannah is only a kid — she rarely gets the ball even to the rim from the top of the key — so there is no way she could make it in the WNBA. So she both could and could not make it. But this cannot be right! What is going on? Here is a plausible explanation: Different sorts of possibility are involved. When we hold fixed current abilities, we are inclined to say there is no way she could make it. When we hold fixed only general physical characteristics, and allow for development, it is natural to say that she might. The scope of what is possible varies with whatever constraints are in play. The weaker the constraints, the broader the range of what is possible.

The sort of possibility we are interested in is *very* broad, and constraints are correspondingly weak. We will allow that a story is *possible* or *consistent* so long as it involves no *internal* contradiction. A story is impossible when it collapses from within. For this it may help to think about the way you respond to ordinary fiction. Consider, say, J.K. Rowling's *Harry Potter and the Prisoner of Azkaban* (much loved by my youngest daughter). Harry and his friend Hermione are at wizarding school. Hermione acquires a "time turner" which allows time travel, and uses it in order to take classes that are offered at the same time. Such devices are no part of the actual world, but they fit into the wizarding world of Harry Potter. So far, then, the story does not contradict itself. So you go along.

At one stage, though, Harry is at a lakeshore under attack by a bunch of fearsome "dementors." His attempts to save himself appear to have failed when a figure across the lake drives the dementors away. But the figure who saves Harry is Harry himself who has come back from the future. Somehow, then, as often happens in these stories, the past depends on the future, at the same time as the future depends on the past: Harry is saved only insofar as he comes back from the future, but he comes back from the future only insofar as he is saved. This, rather than the time travel itself, generates an internal conflict. The story makes it the case that you cannot have Harry's rescue apart from his return, and cannot have Harry's return apart from his rescue. This might make sense if time were always repeating in an eternal loop. But, according to the story, there were times before the rescue and after the return. So the story faces *internal* collapse. Notice: the objection does not have *anything* to do with the way things actually are — with existence of time turners or the like; it has rather to do with the way the story hangs together internally.[1] Similarly, we want to ask whether stories hold together *internally*. If a story holds together internally, it counts for our purposes as consistent and possible. If a story does not hold together, it is not consistent or possible.

In some cases, stories may be consistent with things we know are true in the real world. Thus Hannah could grow up to play in the WNBA. There is nothing about our world that rules this out. But stories may remain consistent though they do not fit with what we know to be true in the real world. Here are cases of time travel and the like. Stories become inconsistent when they collapse internally — as when a story says that some time both can and cannot happen apart from another.

As with a movie or novel, we can say that different things are true or false *in* our

[1] In more consistent cases of time travel (in fiction) time seems to move on different paths so that after yesterday and today, there is *another* yesterday and *another* today. So time does not return to the very point at which it first turns back. In the trouble cases, time seems to move in a sort of "loop" so that a point on the path to today (this very day) goes through tomorrow. With this in mind, it is interesting to think about say, the *Terminator* and *Back to the Future* movies along with, maybe more consistent, Hermione's "academic" travel or *Groundhog Day*. Even if I am wrong, and the Potter story is internally consistent, the overall point should be clear. And it should be clear that I am not saying anything serious about time travel.

stories. In *Harry Potter* it is true that Harry and Hermione travel through time with a timer turner, but false that they go through time in a DeLorean (as in the *Back to the Future* films). In the real world, of course, it is false that there are time turners, and false that DeLoreans go through time. Officially, a complete story is always *maximal* in the sense that *any* sentence is either true or false in it. A story is *inconsistent* when it makes some sentence both true and false. Since, ordinarily, we do not describe every detail of what is true and what is false when we tell a story, what we tell is only part of a maximal story. In practice, however, it will be sufficient for us merely to give or fill in whatever details are relevant in a particular context.

But there are a couple of cases where we cannot say when sentences are true or false in a story. The first is when stories we tell do not fill in relevant details. In *The Wizard of Oz*, it is true that Dorothy wears red shoes. But neither the movie nor the book have anything to say about whether she likes Twinkies. By themselves, then, neither the book nor the movie give us enough information to tell whether "Dorothy likes Twinkies" is true or false in the story. Similarly, there is a problem when stories are inconsistent. Suppose according to some story,

(a) All dogs can fly

(b) Fido is a dog

(c) Fido cannot fly

Given (a), all dogs fly; but from (b) and (c), it seems that not all dogs fly. Given (b), Fido is a dog; but from (a) and (c) it seems that Fido is not a dog. Given (c), Fido cannot fly; but from (a) and (b) it seems that Fido can fly. The problem is not that inconsistent stories say too little, but rather that they say too much. When a story is inconsistent, we will refuse to say that it makes any sentence (simply) true or false.[2]

It will be be helpful to consider some examples of consistent and inconsistent stories:

(a) The real story, "Everything is as it actually is." Since no contradiction is actually true, this story involves no contradiction; so it is internally consistent and possible.

(b) "All dogs can fly: over the years, dogs have developed extraordinarily large and muscular ears; with these ears, dogs can fly." It is bizarre, but not obviously inconsistent. If we allow the consistency of stories according to which monkeys fly,

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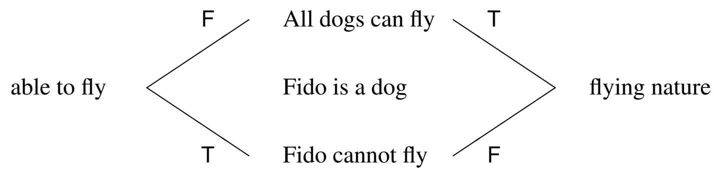
[2]The intuitive picture developed above should be sufficient for our purposes. However, we are on the verge of vexed issues. For further discussion, you may want to check out the vast literature on "possible worlds." Contributions of my own include the introductory article, "Modality," in *The Continuum Companion to Metaphysics*.

as in *The Wizard of Oz*, or elephants fly, as in *Dumbo*, then we should allow that this story is consistent as well.

(c) "All dogs can fly, but my dog Fido cannot; Fido's ear was injured while he was chasing a helicopter, and he cannot fly." This is *not* internally consistent. If all dogs can fly and Fido is a dog, then Fido can fly. You might think that Fido retains a sort of flying nature — just because Fido remains a dog. In evaluating internal consistency, however, we require that *meanings remain the same*.

(C)



If "can fly" means "is able to fly" then in the story it is true that Fido cannot fly, but not true that all dogs can fly (since Fido cannot). If "can fly" means "has a flying nature" then in the story it is true that all dogs can fly, but not true that Fido cannot (because he remains a dog). The only way to keep both 'all dogs fly' and 'Fido cannot fly' true is to *switch* the sense of "can fly" from one use to another. So long as "can fly" means the same in each use, the story is sure to fall apart insofar as it says both that Fido is and is not that sort of thing.

(d) "Germany won WWII; the United States never entered the war; after a long and gallant struggle, England and the rest of Europe surrendered." It did not happen; but the story does not contradict itself. For our purposes, then it counts as possible.

(e) "$1 + 1 = 3$; the numerals '2' and '3' are switched (the numerals are '1', '3', '2', '4', '5', '6', '7'...); so that one and one are three." This story does not hang together. Of course numerals can be switched — so that people would correctly say, '$1 + 1 = 3$'. But this does not make it the case that one and one are three! We tell stories in our own language (imagine that you are describing a foreign-language film in English). Take a language like English except that 'fly' means 'bark'; and consider a movie where dogs are ordinary, so that people in the movie correctly assert, in their language, 'dogs fly'. But changing the words people use to describe a situation does not change the situation. It would be a mistake to tell a friend, in English, that you saw a movie in which there were flying dogs. Similarly, according to our story, people correctly assert, in their language, '$1 + 1 = 3$'. But it is a mistake to say in English (as our story does), that this makes one and one equal to three.

> Some authors prefer talk of "possible worlds," "possible situations" or the like to that of consistent stories. It is conceptually simpler to stick with stories, as I have, than to have situations and distinct descriptions of them. However, it is worth recognizing that our consistent stories are or describe possible situations, so that the one notion matches up directly with the others.

As you approach the following exercises, note that answers to problems indicated by star are provided in the back of the book. It is essential to success that you work a significant body of exercises successfully and independently. So do not neglect exercises!

E1.1. Say whether each of the following stories is internally consistent or inconsistent. In either case, explain why.

*a. Smoking cigarettes greatly increases the risk of lung cancer, although most people who smoke cigarettes do not get lung cancer.

b. Joe is taller than Mary, but Mary is taller than Joe.

*c. Abortion is always morally wrong, though abortion is morally right in order to save a woman's life.

d. Mildred is Dr. Saunders's daughter, although Dr. Saunders is not Mildred's father.

*e. No rabbits are nearsighted, though some rabbits wear glasses.

f. Ray got an 'A' on the final exam in both Phil 200 and Phil 192. But he got a 'C' on the final exam in Phil 192.

*g. Barack Obama was never president of the United States, although Michelle is president right now.

h. Egypt, with about 100 million people is the most populous country in Africa, and Africa contains the most populous country in the world. But the United States has over 200 million people.

*i. The death star is a weapon more powerful than that in any galaxy, though there is, in a galaxy far, far away, a weapon more powerful than it.

j. Luke and the Rebellion valiantly battled the evil Empire, only to be defeated. The story ends there.

E1.2. For each of the following sentences, (i) say whether it is true or false in the real world and then (ii) say, if you can, whether it is true or false according to the accompanying story. In each case, explain your answers. Do not forget about contexts where we refuse to say whether sentences are true or false. The first problem is worked as an example.

a. Sentence: Aaron Burr was never a president of the United States.

   Story: Aaron Burr was the first president of the United States, however he turned traitor and was impeached and then executed.

   (i) It is *true* in the real world that Aaron Burr was never a president of the United States. (ii) But the story makes the sentence *false*, since the story says Burr was the first president.

b. Sentence: In 2006, there were still buffalo.

   Story: A thundering herd of buffalo overran Phoenix, Arizona in early 2006. The city no longer exists.

*c. Sentence: After overrunning Phoenix in early 2006, a herd of buffalo overran Newark, New Jersey.

   Story: A thundering herd of buffalo overran Phoenix, Arizona in early 2006. The city no longer exists.

d. Sentence: There has been an all-out nuclear war.

   Story: After the all-out nuclear war, John Connor organized the Resistance against the machines — who had taken over the world for themselves.

*e. Sentence: Jack Nicholson has swum the Atlantic.

   Story: No human being has swum the Atlantic. Jack Nicholson and Bill Clinton and you are all human beings, and at least one of you swam all the way across!

f. Sentence: Some people have died as a result of nuclear explosions.

   Story: As a result of a nuclear blast that wiped out most of this continent, you have been dead for over a year.

*g. Sentence: Your instructor is not a human being.

   Story: No beings from other planets have ever made it to this country. However, your instructor made it to this country from another planet.

h. Sentence: Lassie is both a television and movie star.

   Story: Dogs have super-big ears and have learned to fly. Indeed, all dogs can fly. Among the many dogs are Lassie and Rin Tin Tin.

*i. Sentence: The Yugo is the most expensive car in the world.

   Story: Jaguar and Rolls Royce are expensive cars. But the Yugo is more expensive than either of them.

j. Sentence: Lassie is a bird who has learned to fly.

   Story: Dogs have super-big ears and have learned to fly. Indeed, all dogs can fly. Among the many dogs are Lassie and Rin Tin Tin.

1.2 The Definitions

The definition of logical validity depends on what is true and false in consistent stories. The definition of soundness builds directly on the definition of validity. Note: in offering these definitions, I *stipulate* the way the terms are to be used; there is no attempt to say how they are used in ordinary conversation; rather, we say what they will mean for us in this context.

LV An argument is *logically valid* if and only if (iff) there is no consistent story in which all the premises are true and the conclusion is false.

LS An argument is *logically sound* iff it is logically valid and all of its premises are true in the real world.

Observe that logical validity has entirely to do with what is true and false in consistent stories. Only with logical soundness is validity combined with premises true in the real world.

Logical (deductive) validity and soundness are to be distinguished from *inductive* validity and soundness or success. For the inductive case, it is natural to focus on the *plausibility* or the *probability* of stories — where an argument is relatively strong when stories that make the premises true and conclusion false are relatively implausible. Logical (deductive) validity and soundness are thus a sort of limiting case, where stories that make premises true and conclusion false are not merely implausible, but impossible. In a deductive argument, conclusions are supposed to be *guaranteed*; in an inductive argument, conclusions are merely supposed to be made probable or plausible. For mathematical logic, we set the inductive case to the side, and focus on the deductive.

Also, do not confuse *truth* with validity and soundness. A sentence is true in the real world when it correctly represents how things are in the real world, and true in a story when it correctly represents how things are in the story. An argument is valid when there is no consistent story that makes the premises true and conclusion false, and sound when it is valid and all its premises are true in the real world. The definitions for validity and soundness depend on truth and falsity for the *premises* and *conclusion* in stories and then in the real world. But, just as it would be a mistake to say that the number three weighs eleven pounds, so truth and falsity do not even apply to arguments themselves, which may be valid or sound.[3]

1.2.1 Invalidity

It will be easiest to begin thinking about *invalidity*. From the definition, if an argument is logically valid, there is no consistent story that makes the premises true and conclusion false. So to show that an argument is invalid, it is enough to *produce* even one consistent story that makes premises true and conclusion false. Perhaps there are stories that result in other combinations of true and false for the premises and conclusion; this does not matter for the definition. However, if there is even one story that makes premises true and conclusion false then, by definition, the argument is not logically valid — and if it is not valid, by definition, it is not logically sound. We can work through this reasoning by means of a simple *invalidity test*. Given an argument, this test has the following four stages.

IT
a. List the premises and negation of the conclusion.
b. Produce a consistent story in which the statements from (a) are all true.
c. Apply the definition of validity.
d. Apply the definition of soundness.

We begin by considering what needs to be done to show invalidity. Then we do it. Finally we apply the definitions to get the results. For a simple example, consider the following argument,

(D)
Eating brussels sprouts results in good health
Ophelia has good health
_____
Ophelia has been eating brussels sprouts

[3] From an introduction to philosophy of language, one might wonder (with good reason) whether the proper bearers of truth are sentences rather than, say, *propositions*. This question is not relevant to the simple point made above.

The definition of validity has to do with whether there are consistent stories in which the premises are true and the conclusion false. Thus, in the first stage, we simply write down what would be the case in a story of this sort.

a. List premises and negation of conclusion.	In any story with the premises true and conclusion false, 1. Eating brussels sprouts results in good health 2. Ophelia has good health 3. Ophelia has not been eating brussels sprouts

Observe that the conclusion is reversed! At this stage we are not giving an argument. Rather we merely list what is the case when the premises are true and conclusion false. Thus there is no line between premises and the last sentence, insofar as there is no suggestion of support. It is easy enough to repeat the premises for (1) and (2). Then for (3) we say what is required for the conclusion to be *false*. Thus, "Ophelia has been eating brussels sprouts" is false if Ophelia has not been eating brussels sprouts. I return to this point below, but that is enough for now.

An argument is invalid if there is even one consistent story that makes the premises true and the conclusion false. Thus, to show invalidity, it is enough to *produce* a consistent story that makes the premises true and conclusion false.

b. Produce a consistent story in which the statements from (a) are all true.	Story: Eating brussels sprouts results in good health, but eating spinach does so as well; Ophelia is in good health but has been eating spinach, not brussels sprouts.

For the statements listed in (a): the story satisfies (1) insofar as eating brussels sprouts results in good health; (2) is satisfied since Ophelia is in good health; and (3) is satisfied since Ophelia has not been eating brussels sprouts. The story *explains* how she manages to maintain her health without eating brussels sprouts, and so the consistency of (1) - (3) together. The story does not have to be true — and, of course, many different stories will do. All that matters is that there is a *consistent* story in which the premises of the original argument are true, and the conclusion is false.

Producing a story that makes the premises true and conclusion false is the creative part. What remains is to apply the definitions of validity and soundness. By LV, an argument is logically valid only if there is no consistent story in which the premises are true and the conclusion is false. So if, as we have demonstrated, there is such a story, the argument cannot be logically valid.

c. Apply the definition of validity.	This is a consistent story that makes the premises true and the conclusion false; thus, by definition, the argument is not logically valid.

By LS, for an argument to be sound, it must have its premises true in the real world *and* be logically valid. Thus if an argument fails to be logically valid, it automatically fails to be logically sound.

d. Apply the definition of soundness.	Since the argument is not logically valid, by definition, it is not logically sound.

Given an argument, the definition of validity depends on stories that make the premises true and the conclusion false. Thus, in step (a) we simply list claims required of any such story. To show invalidity, in step (b), we produce a consistent story that satisfies each of those claims. Then in steps (c) and (d) we apply the definitions to get the final results; for invalidity, these last steps are the same in every case.

It may be helpful to think of stories as a sort of "wedge" to pry the premises of an argument off its conclusion. We pry the premises off the conclusion if there is a consistent way to make the premises true and the conclusion not. If it is possible to insert such a wedge between the premises and conclusion, then a defect is exposed in the way premises are connected to the conclusion. Observe that the flexibility we allow in consistent stories (with flying dogs and the like) corresponds directly to the strength of the required connection between premises and conclusion. If the connection is sufficient to resist all such attempts to wedge the premises off the conclusion, then it is significant indeed. Observe also that our method reflects what we did with argument (A) at the beginning of the chapter: Faced with the premises that only citizens can vote and Hannah is a citizen, it was natural to worry that she might be under-age and so cannot vote. But this is precisely to produce a story that makes the premises true and conclusion false. Thus our method is not "strange" or "foreign"! Rather, it makes rigorous what has seemed natural from the start.

Here is another example of our method. Though the argument may seem on its face not to be a very good one, we can expose its failure by our methods — in fact, again, our method may formalize or make rigorous a way you very naturally think about cases of this sort. Here is the argument,

(E)

I shall run for president

———

I shall be one of the most powerful men on earth

To show that the argument is invalid, we turn to our standard procedure.

a. In any story with the premise true and conclusion false,

   1. I shall run for president
   2. I shall not be one of the most powerful men on earth

b. Story: I do run for president, but get no financing and gain no votes; I lose the election. In the process, I lose my job as a professor and end up begging for scraps outside a Domino's Pizza restaurant. I fail to become one of the most powerful men on earth.

c. This is a consistent story that makes the premise true and the conclusion false; thus, by definition, the argument is not logically valid.

d. Since the argument is not logically valid, by definition, it is not logically sound.

This story forces a wedge between the premise and the conclusion. Thus we use the definition of validity to explain why the conclusion does not properly follow from the premises. It is, perhaps, obvious that *running* for president is not enough to make me one of the most powerful men on earth. Our method forces us to be very explicit about why: running for president leaves open the option of losing, so that the premise does not force the conclusion. Once you get used to it, then, our method may appear as a natural approach to arguments.

If you follow this method for showing invalidity, the place where you are most likely to go wrong is stage (b), telling stories where the premises are true and the conclusion false. Be sure that your story is consistent, and that it verifies *each* of the claims from stage (a). If you do this, you will be fine.

E1.3. Use our invalidity test to show that each of the following arguments is not logically valid, and so not logically sound. Understand terms in their most natural sense.

*a. If Joe works hard, then he will get an 'A'
Joe will get an 'A'
———
Joe works hard

b. Harry had his heart ripped out by a government agent
———
Harry is dead

c. Everyone who loves logic is happy
Jane does not love logic
———
Jane is not happy

d. Our car will not run unless it has gasoline
Our car has gasoline
―――
Our car will run

e. Only citizens can vote
Hannah is a citizen
―――
Hannah can vote

1.2.2 Validity

Suppose I assert that no student at California State University San Bernardino is from Beverly Hills, and attempt to prove it by standing in front of the library and buttonholing students to ask if they are from Beverly Hills — I do this for a week and never find anyone from Beverly Hills. Is the claim that no CSUSB student is from Beverly Hills thereby proved? Of course not – for there may be students I never met. Similarly, failure to find a story to make the premises true and conclusion false does not show that there is not one — for all we know, there might be some story we have not thought of yet. So, to show validity, we need another approach. If we could show that every story which makes the premises true and conclusion false is *inconsistent*, then we could be sure that no *consistent* story makes the premises true and conclusion false — and so, from the definition of validity, we could conclude that the argument is valid. Again, we can work through this by means of a procedure, this time a *validity test*.

VT
a. List the premises and negation of the conclusion.
b. Expose the inconsistency of such a story.
c. Apply the definition of validity.
d. Apply the definition of soundness.

In this case, we begin in just the same way. The key difference arises at stage (b). For an example, consider this argument.

(F)
No car is a person
My mother is a person
―――
My mother is not a car

Since LV has to do with stories where the premises are true and the conclusion false, as before, we begin by listing the premises together with the negation of the conclusion.

a. List premises and negation of conclusion.

In any story with the premises true and conclusion false,

1. No car is a person
2. My mother is a person
3. My mother is a car

Any story where "My mother is not a car" is false, is one where my mother is a car (perhaps along the lines of the 1965 TV series, *My Mother the Car*).

For invalidity, we would produce a consistent story in which (1) - (3) are all true. In this case, to show that the argument is valid, we show that this *cannot* be done. That is, we show that no story that makes each of (1) - (3) true is a consistent story.

b. Expose the inconsistency of such a story.

In any such story,

Given (1) and (3),

4. My mother is not a person

Given (2) and (4),

5. My mother is and is not a person

The reasoning should be clear if you focus *just on the specified lines*. Given (1) and (3), if no car is a person and my mother is a car, then my mother is not a person. But then my mother is a person from (2) and not a person from (4). So we have our goal: any story with (1) - (3) as members contradicts itself and therefore is not consistent. Observe that we could have reached this result in other ways. For example, we might have reasoned from (1) and (2) that (4′), my mother is not a car; and then from (3) and (4′) to the result that (5′) my mother is and is not a car. Either way, an inconsistency is exposed. Thus, as before, there are different options for this creative part.

Now we are ready to apply the definitions of logical validity and soundness. First,

c. Apply the definition of validity.

So no consistent story makes the premises true and conclusion false; so by definition, the argument is logically valid.

For the invalidity test, we produce a consistent story that "hits the target" from stage (a), to show that the argument is invalid. For the validity test, we show that any attempt to hit the target from stage (a) must collapse into inconsistency: no consistent story includes each of the elements from stage (a) so that *there is no consistent story in which the premises are true and the conclusion false*. So by application of LV the argument is logically valid.

Given that the argument is logically valid, LS makes logical soundness depend on whether the premises are true in the real world. Suppose we think the premises of our argument are in fact true. Then,

d. Apply the definition of soundness. In the real world no car is a person and my mother is a person, so all the premises are true; so since the argument is also logically valid, by definition, it is logically sound.

Observe that LS requires for logical soundness that an argument is logically valid and that its *premises* are true in the *real world*. Thus we are no longer thinking about merely possible stories! Soundness depends on the way things are in the real world. And we do not say anything at this stage about claims other than the premises of the original argument! Thus we do not make any claim about the truth or falsity of the conclusion, "my mother is not a car." Rather, the observations have entirely to do with the two premises, "no car is a person" and "my mother is a person." When an argument is valid and the premises are true in the real world, by LS, it is logically sound.

But it will not always be the case that a valid argument has true premises. Say *My Mother the Car* is (surprisingly) a documentary about a person rein*car*nated as a car (the premise of the show) and therefore a true account of some car that is a person. Then some cars are persons and the first premise is false; so you would have to respond as follows,

d′. Since in the real world some cars are persons, the first premise is not true. So, though the argument is logically valid, by definition it is not logically sound.

Another option is that you are in doubt about reincarnation into cars, and in particular about whether some cars are persons. In this case you might respond as follows,

d″. Although in the real world my mother is a person, I cannot say whether no car is a person; so I cannot say whether the first premise is true. So though the argument is logically valid, I cannot say whether it is logically sound.

So once we decide that an argument is valid, for soundness there are three options:

(i) You are in a position to identify all of the premises as true in the real world. In this case, you should do so, and apply the definition for the result that the argument is logically sound.

(ii) You are in a position to say that at least one of the premises is false in the real world. In this case, you should do so, and apply the definition for the result that the argument is not logically sound.

(iii) You cannot identify any premise as false, but neither can you identify them all as true. In this case, you should explain the situation and apply the definition for the result that you are not in a position to say whether the argument is logically sound.

So given a valid argument, there remains a substantive questions about soundness. In some cases, as for example (C) on p. 3, this can be the most controversial part.

Again, given an argument, we say in step (a) what would be the case in any story that makes the premises true and the conclusion false. Then, at step (b), instead of finding a consistent story in which the premises are true and conclusion false, we show that there is no such thing. Steps (c) and (d) apply the definitions for the final results. Observe that only one method can be correctly applied in a given case! If we can produce a consistent story according to which the premises are true and the conclusion is false, then it is not the case that no consistent story makes the premises true and the conclusion false. Similarly, if no consistent story makes the premises true and the conclusion false, then we will not be able to produce a consistent story that makes the premises true and the conclusion false.

For showing validity, the most difficult steps are (a) and (b), where we say what happens in every story where the premises true and the conclusion false. For an example, consider the following argument.

(G)

All collies can fly
All collies are dogs
———
All dogs can fly

It is invalid. We can easily tell a story that makes the premises true and the conclusion false — say one where collies fly but dachshunds do not. Suppose, however, that we proceed with the validity test as follows,

a. In any story with the premises true and conclusion false,

   1. All collies can fly
   2. All collies are dogs
   3. No dogs can fly

b. In any such story,

   Given (1) and (2),
   4. Some dogs can fly
   Given (3) and (4),
   5. Some dogs can and cannot fly

c. So no consistent story makes the premises true and conclusion false; so by definition, the argument is logically valid.

d. Since in the real world collies cannot fly, the first premise is not true. So, though the argument is logically valid, by definition it is not logically sound.

The reasoning at (b), (c) and (d) is correct. Any story with (1) - (3) is inconsistent. But something is wrong. (Can you see what?) There is a mistake at (a): It is not the case that every story that makes the premises true and conclusion false includes (3). The negation of "All dogs can fly" is not "No dogs can fly," but rather, "Not all dogs can fly" (or "Some dogs cannot fly"). All it takes to falsify the claim that *all dogs fly* is some dog that does not. Thus, for example, all it takes to falsify the claim that everyone will get an 'A' is one person who does not (on this, see the extended discussion on p. 22). So for argument (G) we have indeed shown that every story of a certain sort is inconsistent, but have not shown that every story which makes the premises true and conclusion false is inconsistent. In fact, as we have seen, there are consistent stories that make the premises true and conclusion false.

Similarly, in step (b) it is easy to get confused if you consider too much information at once. Ordinarily, if you focus on sentences singly or in pairs, it will be clear what must be the case in every story including those sentences. It does not matter which sentences you consider in what order, so long as in the end, you reach a contradiction according to which something is and is not so.

So far, we have seen our procedures applied in contexts where it is given ahead of time whether an argument is valid or invalid. But not all situations are so simple. In the ordinary case, it is not given whether an argument is valid or invalid. In this case, there is no magic way to say ahead of time which of our two tests, IT or VT applies. The only thing to do is to try one way — if it works, fine. If it does not, try the other. It is perhaps most natural to begin by looking for stories to pry the premises off the conclusion. If you can find a consistent story to make the premises true and conclusion false, the argument is invalid. If you cannot find any such story, you may begin to suspect that the argument is valid. This suspicion does not itself amount to a demonstration of validity! But you might try to turn your suspicion into such a demonstration by attempting the validity method. Again, if one procedure works, the other better not!

E1.4. Use our validity procedure to show that each of the following is logically valid, and decide (if you can) whether it is logically sound.

Negation and Quantity

In general you want to be careful about negations. To negate any claim $\mathscr{P}$ it is always correct to write simply, *it is not the case that* $\mathscr{P}$. You may choose to do this for conclusions in the first step of our procedures. At some stage, however, you will need to understand what the negation comes to. We have chosen to offer interpreted versions in the text. It is easy enough to see that,

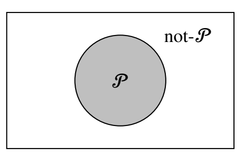
My mother is a car and My mother is not a car

negate one another. However, there are cases where caution is required. This is particularly the case with terms involving quantities.

Say the conclusion of your argument is, ‘there are at least ten apples in the basket’. Clearly a story according to which there are, say, three apples in the basket makes this conclusion false. However, there are other ways to make the conclusion false — as if there are two apples or seven. Any of these are fine for showing invalidity.

But when you show that an argument is valid, you must show that *any* story that makes the premises true and conclusion false is inconsistent. So it is not sufficient to show that stories with (the premises true and) three apples in the basket contradict. Rather, you need to show that any story that includes the premises and *less than ten* apples fails. Thus in step (a) of our procedure we always say what is so in *every* story that makes the premises true and conclusion false. So, in (a) you would have the premises and, ‘there are less than ten apples in the basket’.

If a statement is included in some range of consistent stories, then its negation says what is so in all the others — all the ones where it is not so.



all consistent stories

That is why the negation of ‘there are at least ten’ is ‘there are less than ten’.

The same point applies with other quantities. Consider some grade examples: First, if a professor says, “everyone will not get an ‘A’,” she says something disastrous — nobody in your class will get an ‘A’. In order too deny it, to show that she is wrong, all you need is at least one person that gets an ‘A’. In contrast, if she says, “someone will not get an ‘A’,” she says only what you expect from the start — that not everyone will get an ‘A’. To deny this, you would need that everyone gets an ‘A’. Thus the following pairs negate one another.

Everyone will not get an ‘A’ and Someone will get an ‘A’

Someone will not get an ‘A’ and Everyone will get an ‘A’

It is difficult to give rules to cover all the cases. The best is just to think about what you are saying, perhaps with reference to examples like these.

*a. If Bill is president, then Hillary is first lady
Hillary is not first lady
———
Bill is not president

b. Only fools find love
Elvis was no fool
———
Elvis did not find love

c. If there is a good and omnipotent god, then there is no evil
There is evil
———
There is no good and omnipotent god

d. All sparrows are birds
All birds fly
———
All sparrows fly

e. All citizens can vote
Hannah is a citizen
———
Hannah can vote

E1.5. Use our procedures to say whether the following are logically valid or invalid, and sound or unsound. Hint: You may have to do some experimenting to decide whether the arguments are logically valid or invalid — and so decide which procedure applies.

a. If Bill is president, then Hillary is first lady
Bill is president
———
Hillary is first lady

b. Most professors are insane
TR is a professor
———
TR is insane

*c. Some dogs have red hair
Some dogs have long hair
———
Some dogs have long, red hair

d. If you do not strike the match, then it does not light
The match lights
———
You strike the match

e. Shaq is taller than Kobe
Kobe is at least as tall as TR
———
Kobe is taller than TR

1.3 Some Consequences

We now know what logical validity and soundness are, and should be able to identify them in simple cases. Still, it is one thing to know what validity and soundness are, and another to know why they matter. So in this section I turn to some consequences of the definitions.

1.3.1 Soundness and Truth

First, a consequence we want: The conclusion of every sound argument is true in the real world. Observe that this is *not* part of what we require to show that an argument is sound. LS requires just that an argument is valid and that its *premises* are true. However it is a consequence of validity plus true premises that the conclusion is true as well.

$$\text{sound} \quad \Longrightarrow \quad \frac{\text{valid} + \text{true premises}}{\text{true conclusion}}$$

To see this, consider a two-premise argument. Say the *real* story describes the real world; so the sentences of the real story are all true in the real world. Then in the real story, the premises and conclusion of our argument must fall into one of the following combinations of true and false:

1	2	3	4	5	6	7	8	
T	T	T	F	T	F	F	F	combinations for
T	T	F	T	F	T	F	F	the real story
T	F	T	T	F	F	T	F	

These are all the combinations of T and F. Say the argument is logically valid; then no consistent story makes the premises true and the conclusion false. But the real story is a consistent story. So we can be sure that the real story does not result in combination (2). So far, the real story might result in any of the other combinations. Thus the conclusion of a valid argument may or may not be true in the real world. Now say the argument is sound; then it is valid and all its premises are true in the real world. Again, since it is valid, the real story does not result in combination (2). And since the premises of a sound argument are true in the real world, we can be sure that the premises do not fall into any of the combinations (3) - (8). (1) is the only combination left: in the real story, and so in the real world, the conclusion of a sound argument is true. And not only in this case but in general, if an argument is sound its conclusion is true in the real world: Since a sound argument is valid, there is no consistent story where its premises are true and the conclusion is false, and since the premises really are true, the conclusion has to be true as well. Put another way, if an argument is sound, its premises are true in the real story; but then if the conclusion is false, the real story has the premises true and conclusion false — and the argument is not valid. So if an argument is sound, if it is valid and its premises are true, its conclusion must be true.

Note again: we do not need that the conclusion is true in the real world in order to *decide* that an argument is sound; saying that the conclusion is true is no part of our procedure for validity or soundness! Rather, by discovering that an argument is logically valid and that its *premises* are true, we *establish* that it is sound; this gives us the result that its conclusion therefore is true. And that is just what we want.

1.3.2 Validity and Form

It is worth observing a connection between what we have done and argument form. Some of the arguments we have seen so far are of the same general *form*. Thus both of the arguments on the left have the form on the right.

(H)	If Joe works hard, then he will get an 'A' Joe works hard ——— Joe will get an 'A'	If Hannah is a citizen then she can vote Hannah is a citizen ——— Hannah can vote	If $\mathcal{P}$ then $\mathcal{Q}$ $\mathcal{P}$ ——— $\mathcal{Q}$

As it turns out, all arguments of this form are valid. In contrast, the following arguments with the indicated form are not.

(I)	If Joe works hard then he will get an 'A' Joe will get an 'A' ——— Joe works hard	If Hannah can vote, then she is a citizen Hannah is a citizen ——— Hannah can vote	If $\mathcal{P}$ then $\mathcal{Q}$ $\mathcal{Q}$ ——— $\mathcal{P}$

There are stories where, say, Joe cheats for the 'A', or Hannah is a citizen but not old enough to vote. In these cases, there is some way to obtain condition $\mathscr{Q}$ other than by having $\mathscr{P}$ — this is what the stories bring out. And, generally, it is often possible to characterize arguments by their forms, where a form is *valid* iff every instance of it is logically valid. Thus the first form listed above is valid, and the second not.

In chapters to come, we take advantage of certain very general formal or structural features of arguments to identify ones that are valid and ones that are invalid. For now, though, it is worth noting that some presentations of critical reasoning (which you may or may not have encountered), take advantage of patterns like those above, listing typical ones that are valid, and typical ones that are not (for example, Cederblom and Paulsen, *Critical Reasoning*). A student may then identify valid and invalid arguments insofar as they match the listed forms. This approach has the advantage of simplicity — and one may go quickly to applications of the logical notions for concrete cases. But the approach is limited to application of listed forms, and so to a very narrow range of arguments. In contrast, our approach based on definition LV has application to arbitrary arguments. Further, a mere listing of valid forms does not explain their relation to truth, where the definition is directly connected. Finally, for our logical machine, within a certain range we shall develop develop an account of validity for quite arbitrary forms. So we are pursuing a general account or theory of validity that goes well beyond the mere lists of these other more traditional approaches.[4]

1.3.3 Relevance

Another consequence seems less welcome. Consider the following argument.

(J)
Snow is white
Snow is not white
———
All dogs can fly

It is natural to think that the premises are not connected to the conclusion in the right way — for the premises have nothing to do with the conclusion — and that this argument therefore should not be logically valid. But if it is not valid, by definition, there is a consistent story that makes the premises true and the conclusion false. And, in this case, there is no such story, for *no consistent story makes the premises true*.

[4] Some authors introduce a notion of *formal validity* (maybe in the place of logical validity as above) such that an argument is formally valid iff it has some valid form. As above, formal validity is parasitic on logical validity, together with a to-be-specified notion of form. But if an argument is formally valid, it is logically valid. So if our logical machine is adequate to identify formal validity, it identifies logical validity as well.

Thus, by definition, this argument is logically valid. The procedure applies in a straightforward way. Thus,

a. In any story that makes the premises true and conclusion false,

   1. Snow is white
   2. Snow is not white
   3. Some dogs cannot fly

b. In any such story,

   Given (1) and (2),
   4. Snow is and is not white

c. So no consistent story makes the premises true and conclusion false; so by definition, the argument is logically valid.

d. Since in the real world snow is white, the second premise is not true. So, though the argument is logically valid, by definition it is not logically sound.

This seems bad! Intuitively, there is something wrong with the argument. But, on our official definition, it is logically valid. One might rest content with the observation that, even though the argument is logically valid, it is not logically sound. But this does not remove the general worry. For this argument,

(K)
There are fish in the sea
———
Nothing is round and not round

has all the problems of the other and is logically *sound* as well. (Why?) One might, on the basis of examples of this sort, decide to reject the (classical) account of validity with which we have been working. Some do just this.[5] But, for now, let us see what can be said in defense of the classical approach. (And the classical approach is, no doubt, the approach you have seen or will see in any standard course on critical thinking or logic.)

As a first line of defense, one might observe that the conclusion of every sound argument is true and ask, "What more do you want?" We use arguments to demonstrate the truth of conclusions. And nothing we have said suggests that sound arguments

[5]Especially the so-called "relevance" logicians. For an introduction, see Graham Priest, *Non-Classical Logics*. But his text presumes mastery of material corresponding to Part I and Part II (or at least Part I with chapter 7) of this one. So the non-classical approaches develop or build on the classical one developed here.

do not have true conclusions: An argument whose premises are inconsistent is sure to be unsound. And an argument whose conclusion cannot be false is sure to have a true conclusion. So soundness may seem sufficient for our purposes. Even though we accept that there remains something about argument goodness that soundness leaves behind, we can insist that soundness is useful as an intellectual tool. Whenever it is the truth or falsity of a conclusion that matters, we can profitably employ the classical notions.

But one might go further, and dispute even the suggestion that there is something about argument goodness that soundness leaves behind. Consider the following two argument forms.

$$\text{(ds)}\quad \frac{\mathcal{P} \text{ or } \mathcal{Q},\ \text{not-}\mathcal{P}}{\mathcal{Q}} \qquad\qquad \text{(add)}\quad \frac{\mathcal{P}}{\mathcal{P} \text{ or } \mathcal{Q}}$$

According to ds (*disjunctive syllogism*), if you are given that $\mathcal{P}$ or $\mathcal{Q}$ and that not-$\mathcal{P}$, you can conclude that $\mathcal{Q}$. If you have cake or ice cream, and you do not have cake, you have ice cream; if you are in California or New York, and you are not in California, you are in New York; and so forth. Thus ds seems hard to deny. And similarly for add (*addition*). Where 'or' means 'one or the other or both', when you are given that $\mathcal{P}$, you can be sure that $\mathcal{P}$ or anything. Say you have cake, then you have cake or ice cream, cake or brussels sprouts, and so forth; if grass is green, then grass is green or pigs have wings, grass is green or dogs fly, and so forth.

Return now to our problematic argument. As we have seen, it is valid according to the classical definition LV. We get a similar result when we apply the ds and add principles.

1.	Snow is white	premise
2.	Snow is not white	premise
3.	Snow is white or all dogs can fly	from 1 and add
4.	All dogs can fly	from 2 and 3 and ds

If snow is white, then snow is white or anything. So snow is white or dogs fly. So we use line 1 with add to get line 3. But if snow is white or dogs fly, and snow is not white, then dogs fly. So we use lines 2 and 3 with ds to reach the final result. So our principles ds and add go hand-in-hand with the classical definition of validity. The argument is valid on the classical account; and with these principles, we can move from the premises to the conclusion. If we want to reject the validity of this argument, we will have to reject not only the classical notion of validity, but also one of our principles ds or add. And it is not obvious that one of the principles should go. If we decide to retain both ds and add then, seemingly, the classical definition of validity should stay as well. If we have intuitions according to which ds and add should stay, and also that the definition of validity should go, we have conflicting intuitions. Thus our intuitions might, at least, be sensibly resolved in the classical direction.

These issues are complex, and a subject for further discussion. For now, it is enough for us to treat the classical approach as a useful tool: It is useful in contexts where what we care about is whether conclusions are true. And alternate approaches to validity typically develop or modify the classical approach. So it is natural to begin where we are, with the classical account. At any rate, this discussion constitutes a sort of acid test: If you understand the validity of the "snow is white" and "fish in the sea" arguments (J) and (K), you are doing well — you understand *how* the definition of validity works, with its results that may or may not now seem controversial. If you do not see what is going on in those cases, then you have not yet understood how the definitions work and should return to section 1.2 with these cases in mind.

E1.6. Use our procedures to say whether the following are logically valid or invalid, and sound or unsound. Hint: You may have to do some experimenting to decide whether the arguments are logically valid or invalid — and so decide which procedure applies.

a. Bob is over six feet tall
   Bob is under six feet tall
   ———
   Bob is disfigured

b. Marilyn is not over six feet tall
   Marilyn is not under six feet tall
   ———
   Marilyn is not in the WNBA

c. There are fish in the sea
   ———
   Nothing is round and not round

*d. Cheerios are square
   Chex are round
   ———
   There is no round square

e. All dogs can fly
   Fido is a dog
   Fido cannot fly
   ———
   I am blessed

E1.7. Respond to each of the following.

a. Create another argument of the same form as the first set of examples (H) from section 1.3.2, and then use our regular procedures to decide whether it is logically valid and sound. Is the result what you expect? Explain.

b. Create another argument of the same form as the second set of examples (I) from section 1.3.2, and then use our regular procedures to decide whether it is logically valid and sound. Is the result what you expect? Explain.

E1.8. Which of the following are true, and which are false? In each case, explain your answers, with reference to the relevant definitions. The first is worked as an example.

a. A logically valid argument is always logically sound.

False. An argument is sound iff it is logically valid and all of its premises are true in the real world. Thus an argument might be valid but fail to be sound if one or more of its premises is false in the real world.

b. A logically sound argument is always logically valid.

*c. If the conclusion of an argument is true in the real world, then the argument must be logically valid.

d. If the premises and conclusion of an argument are true in the real world, then the argument must be logically sound.

*e. If a premise of an argument is false in the real world, then the argument cannot be logically valid.

f. If an argument is logically valid, then its conclusion is true in the real world.

*g. If an argument is logically sound, then its conclusion is true in the real world.

h. If an argument has contradictory premises (its premises are true in no consistent story), then it cannot be logically valid.

*i. If the conclusion of an argument cannot be false (is false in no consistent story), then the argument is logically valid.

j. The premises of every logically valid argument are relevant to its conclusion.

E1.9. For each of the following concepts, explain in an essay of about two pages, so that (high-school age) Hannah could understand. In your essay, you should (i) identify the objects to which the concept applies, (ii) give and explain the definition, and give and explicate examples of your own construction (iii) where the concept applies, and (iv) where it does not. Your essay should exhibit an understanding of methods from the text.

a. Logical validity

b. Logical soundness

E1.10. Do you think we should accept the classical account of validity? In an essay of about two pages, explain your position, with special reference to difficulties raised in section 1.3.3.

Chapter 2

Formal Languages

Having said in chapter 1 what validity and soundness are, we now turn to our logical machine. As depicted in the picture of elements for symbolic logic on p. 2, this machine begins with symbolic representations of ordinary reasoning. In this chapter we introduce the formal languages by introducing their *grammar*. After some brief introductory remarks, the chapter divides into sections that introduce grammar for a *sentential* language $\mathcal{L}_s$ (section 2.2), and then the grammar for an extended *quantificational* language $\mathcal{L}_q$ (section 2.3).

2.1 Introductory

There are different ways to introduce a formal language. It is natural to introduce expressions of a new language in relation to expressions of one that is already familiar. Thus, a standard course in a foreign language is likely to present vocabulary lists of the sort,

chou:	cabbage
petit:	small
⋮	

But the terms of a foreign language are not *originally defined* by such lists. Rather French, in this case, has conventions of its own such that sometimes '*chou*' corresponds to 'cabbage' and sometimes it does not. It is not a legitimate criticism of a Frenchman who refers to his sweetheart as *mon petit chou* to observe that she is no cabbage! Though it is possible to use such correlations to introduce the conventions of a new language, it is also possible to introduce a language "as itself" — the way a native speaker learns it. In this case, one avoids the danger of importing conventions and patterns from one language onto the other. Similarly, the expressions of a formal language might be introduced in correlation with expressions of, say, English. But this runs the risk of obscuring just what the official definitions accomplish. Since we will be concerned extensively with what follows from the definitions, it is best to introduce our languages in their "pure" forms.

In this chapter, we develop the *grammar* of our formal languages. Consider the following algebraic expressions,

$$a + b = c \qquad\qquad a + \; = \; c$$

Until we know what numbers are assigned to the terms (as $a = 1, b = 2, c = 3$), we cannot evaluate the first for truth or falsity. Still, we can confidently say that it is grammatical where the other is not. We shall be able to evaluate the grammar of formal languages in a similar way. Though, eventually, our goal is to represent ordinary reasonings in a formal language, we do not have to know what the language represents in order to decide if a sentence is grammatically correct. Again, just as a computer can check the spelling and grammar of English without reference to meaning, so we can introduce the vocabulary and grammar of our formal languages without reference to what their expressions mean or what makes them true. The grammar, taken alone, is completely straightforward. Taken this way, we work directly from the definitions, without "pollution" from associations with English or whatever.

So we want the definitions. Even so, it may be helpful to offer some hints that foreshadow how things will go. Do not take these as defining anything! Still, it is nice to have a sense of how it fits together. Consider some simple sentences of an ordinary language, say, 'The butler is guilty' and 'The maid is guilty'. It will be convenient to introduce capital letters corresponding to these, say, B and M. Such sentences may combine to form ones that are more complex as, '*It is not the case that* the butler is guilty' or '*If* the butler is guilty, *then* the maid is guilty'. We shall find it convenient to express these, '$\sim$the butler is guilty' and 'The butler is guilty $\rightarrow$ the maid is guilty', with operators $\sim$ and $\rightarrow$. Putting these together we get, $\sim B$ and $B \rightarrow M$. Operators may be combined in obvious ways so that $B \rightarrow \sim M$ says that if the butler is guilty then the maid is not. And so forth. We shall see that incredibly complex expressions of this sort are possible!

In this case, simple sentences, 'The butler is guilty' and 'The maid is guilty' are "atoms" and complex sentences are built out of them. This is characteristic of the *sentential* languages to be considered in section 2.2. For the *quantificational* languages of section 2.3, certain sentence *parts* are taken as atoms. So quantificational languages expose structure beyond that for the sentential case. Perhaps, though, this will be enough to give you a glimpse of the overall strategy and aims for the formal languages of which we are about to introduce the grammar.

2.2 Sentential Languages

Just as algebra or English have their own vocabulary or symbols, and then grammatical rules for the way the vocabulary is combined, so our formal language has its own vocabulary, and then grammatical rules for the way the vocabulary is combined. In this section, we introduce the vocabulary for a sentential language, introduce the grammatical rules, and conclude with some discussion of abbreviations for official expressions.

2.2.1 Vocabulary

We begin, then, with the vocabulary. In this section, we say which symbols are included in the language, and introduce some conventions for talking about the symbols.

In the sentential case, vocabulary includes,

VC (p) Punctuation symbols: ( )

(o) Operator symbols: $\sim$ $\rightarrow$

(s) A non-empty countable collection of sentence letters

And that is all. $\sim$ is *tilde* and $\rightarrow$ is *arrow*.[1] In order to fully specify the vocabulary of any particular sentential language, we need to identify its sentence letters — so far as definition VC goes, different languages may differ in their collections of sentence letters. The only constraint on such specifications is that the collections of sentence letters be non-empty and countable. A collection is *non-empty* iff it has at least one member. So any sentential language has at least one sentence letter. A collection is *countable* iff its members can be matched one-to-one with all (or some) of the integers. Thus we might let the sentence letters be $A, B \ldots Z$, where these correlate with the integers $1 \ldots 26$. Or we might let there be infinitely many sentence letters, $S_0, S_1, S_2 \ldots$ where the letters are correlated with the integers by their subscripts.

[1] Sometimes sentential languages are introduced with different symbols, for example, $\neg$ for $\sim$, or $\supset$ for $\rightarrow$. It should be easy to convert between presentations of the different sorts. And sometimes sentential languages include operators in addition to $\sim$ and $\rightarrow$ (for example, $\vee$, $\wedge$, $\leftrightarrow$). Such symbols will be introduced in due time — but as abbreviations for complex official expressions.

So there is room for different sentential languages. Having made this point, though, we immediately focus on a standard sentential language $\mathcal{L}_s$ whose sentence letters are Roman italics $A \dots Z$ with or without integer subscripts. Thus,

$$A \qquad B \qquad K \qquad Z$$

are sentence letters of $\mathcal{L}_s$. Similarly,

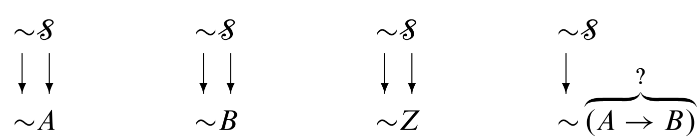
$$A_1 \qquad B_3 \qquad K_7 \qquad Z_{23}$$

are sentence letters of $\mathcal{L}_s$. We will not use the subscripts very often, but they guarantee that we never run out of sentence letters! Perhaps surprisingly, as described in the box on p. 36 (and E2.2), these letters too can be correlated with the integers. Official sentences of $\mathcal{L}_s$ are built out of this vocabulary.

To proceed, we need some conventions for talking *about* expressions of a language like $\mathcal{L}_s$. Here, $\mathcal{L}_s$ is an *object* language — the thing we want to talk about, and we require conventions for the *metalanguage* — for talking about the object language. In general, for any formal object language $\mathcal{L}$, an *expression* is a sequence of one or more elements of its vocabulary. Thus $(A \rightarrow B)$ is an expression of $\mathcal{L}_s$, but $(A \star B)$ is not. (What is the difference?) We shall use script characters $\mathcal{A} \dots \mathcal{Z}$ as variables that range over expressions. '$\sim$', '$\rightarrow$', '(', and ')' represent themselves. Concatenated or joined symbols in the metalanguage represent the concatenation of the symbols they represent.

To see how this works, think of metalinguistic expressions as "mapping" to object-language ones. Thus, for example, where $\mathcal{S}$ represents an arbitrary sentence letter, $\sim\mathcal{S}$ may represent any of, $\sim A$, $\sim B$, or $\sim Z$. But $\sim(A \rightarrow B)$ is not of that form, for it does not consist of a tilde followed by a sentence letter. With $\mathcal{S}$ restricted to sentence letters, there is a straightforward map from $\sim\mathcal{S}$ onto $\sim A$, $\sim B$, or $\sim Z$, but not from $\sim\mathcal{S}$ onto $\sim(A \rightarrow B)$.

(A)



In the first three cases, $\sim$ maps to itself, and $\mathcal{S}$ to a sentence letter. In the last case there is no map. We might try mapping $\mathcal{S}$ to A or B; but this would leave the rest of the expression unmatched. While $\sim(A \rightarrow B)$ is not of the form $\sim\mathcal{S}$, if we let $\mathcal{P}$ represent any arbitrary expression, then $\sim(A \rightarrow B)$ is of the form $\sim\mathcal{P}$, for it consists of a tilde followed by an expression of some sort. An object-language expression has some metalinguistic form just when there is a complete map from the metalinguistic form to it.

Countability

To see the full range of languages which are allowed under VC, observe how multiple infinite series of sentence letters may satisfy the countability constraint. Thus, for example, suppose we have two series of sentence letters, $A_0, A_1 \ldots$ and $B_0, B_1 \ldots$ These can be correlated with the integers as follows,

$$\begin{array}{cccccccc} A_0 & B_0 & A_1 & B_1 & A_2 & B_2 & \\ | & | & | & | & | & | & \ldots \\ 0 & 1 & 2 & 3 & 4 & 5 & \end{array}$$

For any integer n, A_n is matched with $2n$, and B_n with $2n + 1$. So each sentence letter is matched with some integer; so the sentence letters are countable. If there are three series, they may be correlated,

$$\begin{array}{cccccccc} A_0 & B_0 & C_0 & A_1 & B_1 & C_1 & \\ | & | & | & | & | & | & \ldots \\ 0 & 1 & 2 & 3 & 4 & 5 & \end{array}$$

so that every sentence letter is matched to some integer. And similarly for any finite number of series. And there might be 26 such series, as for our language $\mathscr{L}_s$.

In fact even this is not the most general case. If there are *infinitely* many series of sentence letters, we can still line them up and correlate them with the integers. Here is one way to proceed. Order the letters as follows,

$$\begin{array}{cccccccc} A_0 & \rightarrow & A_1 & & A_2 & \rightarrow & A_3 & \ldots \\ & \swarrow & & \nearrow & & \swarrow & & \\ B_0 & & B_1 & & B_2 & & B_3 & \ldots \\ \downarrow & \nearrow & & \swarrow & & & & \\ C_0 & & C_1 & & C_2 & & C_3 & \ldots \\ & \swarrow & & & & & & \\ D_0 & & D_1 & & D_2 & & D_3 & \ldots \\ \vdots & & & & & & & \end{array}$$

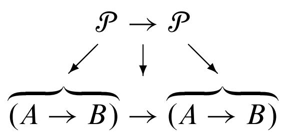
And following the arrows, match them accordingly with the integers,

$$\begin{array}{cccccccc} A_0 & A_1 & B_0 & C_0 & B_1 & A_2 & \\ | & | & | & | & | & | & \ldots \\ 0 & 1 & 2 & 3 & 4 & 5 & \end{array}$$

so that, again, any sentence letter is matched with some integer. It may seem odd that we can line symbols up like this, but it is hard to dispute that we have done so. Thus we may say that VC is compatible with a wide variety of specifications, but also that all legitimate specifications have something in common: If a collection is countable, it is possible to sort its members into a series with a first member, a second member, and so forth.

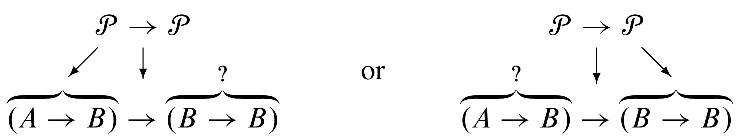
Say $\mathcal{P}$ represents any arbitrary expression. Then by similar reasoning, $(A \rightarrow B) \rightarrow (A \rightarrow B)$ is of the form $\mathcal{P} \rightarrow \mathcal{P}$.

(B)



In this case, $\mathcal{P}$ maps to all of $(A \rightarrow B)$ and $\rightarrow$ to itself. A constraint on our maps is that the use of the metavariables $\mathcal{A} \dots \mathcal{Z}$ must be consistent within a given map. Thus $(A \rightarrow B) \rightarrow (B \rightarrow B)$ is not of the form $\mathcal{P} \rightarrow \mathcal{P}$.

(C)



We are free to associate $\mathcal{P}$ with whatever we want. However, within a given map, once $\mathcal{P}$ is associated with some expression, we have to use it consistently within that map.

Observe again that $\sim\mathcal{S}$ and $\mathcal{P} \rightarrow \mathcal{P}$ are not expressions of $\mathcal{L}_s$. Rather, we use them to talk about expressions of $\mathcal{L}_s$. And it is important to see how we can use the metalanguage to make claims about a range of expressions all at once. Given that $\sim A$, $\sim B$ and $\sim Z$ are all of the form $\sim\mathcal{S}$, when we make some claim about expressions of the form $\sim\mathcal{S}$, we say something about each of them — but not about $\sim(A \rightarrow B)$. Similarly, if we make some claim about expressions of the form $\mathcal{P} \rightarrow \mathcal{P}$, we say something with application to a *range* of expressions. In the next section, for the specification of *formulas*, we use the metalanguage in just this way.

E2.1. Assuming that $\mathcal{S}$ may represent any sentence letter, and $\mathcal{P}$ any arbitrary expression of $\mathcal{L}_s$, use maps to determine whether each of the following expressions is (i) of the form $(\mathcal{S} \rightarrow \sim\mathcal{P})$ and then (ii) whether it is of the form $(\mathcal{P} \rightarrow \sim\mathcal{P})$. In each case, explain your answers.

a. $(A \rightarrow \sim A)$

b. $(A \rightarrow \sim(R \rightarrow \sim Z))$

c. $(\sim A \rightarrow \sim(R \rightarrow \sim Z))$

d. $((R \rightarrow \sim Z) \rightarrow \sim(R \rightarrow \sim Z))$

*e. $((\rightarrow \sim) \rightarrow \sim(\rightarrow \sim))$

E2.2. On the pattern of examples from the countability guide on p. 36, show that the sentence letters of $\mathcal{L}_s$ are countable — that is, that they can be correlated with the integers. On the scheme you produce, what integers correlate with A, B_1 and C_{10}? Hint: Supposing that A without subscript is like A_0, for any integer n, you should be able to produce a formula for the position of any A_n, and similarly for B_n, C_n and the like. Then it will be easy to find the position of any letter, even if the question is about, say, L_{125}.

2.2.2 Formulas

We are now in a position to say which expressions of a sentential language are its grammatical *formulas* and *sentences*. The specification itself is easy. We will spend a bit more time explaining how it works. For a given sentential language $\mathcal{L}$,

FR (s) If $\mathscr{S}$ is a sentence letter, then $\mathscr{S}$ is a *formula*.

($\sim$) If $\mathscr{P}$ is a formula, then $\sim\mathscr{P}$ is a *formula*.

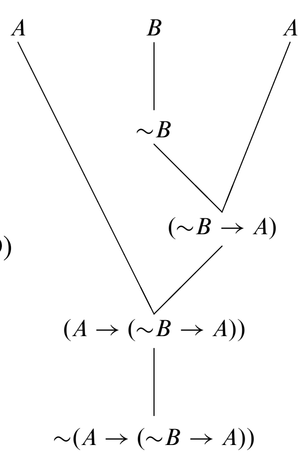
($\rightarrow$) If $\mathscr{P}$ and $\mathscr{Q}$ are formulas, then $(\mathscr{P} \rightarrow \mathscr{Q})$ is a *formula*.

(CL) Any formula may be formed by repeated application of these rules.

At this stage, we simply identify formulas and sentences. For any sentential language $\mathcal{L}$, an expression is a *sentence* iff it is a formula.

FR is a first example of a *recursive* definition. Such definitions always build from the parts to the whole. Frequently we can use "tree" diagrams to see how they work. Thus, for example, by repeated applications of the definition, $\sim(A \rightarrow (\sim B \rightarrow A))$ is a formula and sentence of $\mathcal{L}_s$.

(D)



These are formulas by FR(s)

Since B is a formula, this is a formula by FR($\sim$)

Since $\sim B$ and A are formulas, this is a formula by FR($\rightarrow$)

Since A and $(\sim B \rightarrow A)$ are formulas, this is a formula by FR($\rightarrow$)

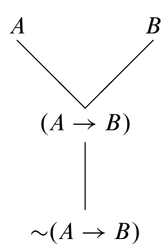
Since $(A \rightarrow (\sim B \rightarrow A))$ is a formula, this is a formula by FR($\sim$)

By FR(s), the sentence letters, A, B and A are formulas; given this, clauses FR($\sim$) and FR($\rightarrow$) let us conclude that other, more complex, expressions are formulas as well. Notice that, in the definition, $\mathcal{P}$ and $\mathcal{Q}$ may be any expressions that are formulas: By FR($\sim$), if B is a formula, then tilde followed by B is a formula; but similarly, if $\sim B$ and A are formulas, then an opening parenthesis followed by $\sim B$, followed by $\rightarrow$ followed by A and then a closing parenthesis is a formula; and so forth as on the tree above. You should follow through each step very carefully. In contrast, $(A{\sim}B)$ for example, is not a formula. A is a formula and $\sim B$ is a formula; but there is no way to put them together, by the definition, without $\rightarrow$ in between.

A recursive definition always involves some "basic" starting elements, in this case, sentence letters. These occur across the top row of our tree. Other elements are constructed, by the definition, out of ones that come before. The last, *closure*, clause tells us that any formula is built this way. To demonstrate that an expression is a formula and a sentence, it is sufficient to construct it, according to the definition, on a tree. If an expression is not a formula, there will be no way to construct it according to the rules.

Here are a couple of last examples which emphasize the point that *you must maintain and respect parentheses* in the way you construct a formula. Thus consider,

(E)



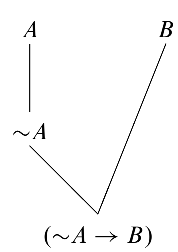
These are formulas by FR(s)

Since A and B are formulas, this is a formula by FR($\rightarrow$)

Since $(A \rightarrow B)$ is a formula, this is a formula by FR($\sim$)

And compare it with,

(F)



These are formulas by FR(s)

Since A is a formula, this is a formula by FR($\sim$)

Since $\sim A$ and B are formulas, this is a formula by FR($\rightarrow$)

Once you have $(A \rightarrow B)$ as in the first case, the only way to apply FR($\sim$) puts the tilde on the outside. To get the tilde inside the parentheses, by the rules, it has to go on first, as in the second case. The significance of this point emerges immediately below.

It will be helpful to have some additional definitions, each of which may be introduced in relation to the trees. First, for any formula $\mathcal{P}$, each formula which appears in the tree for $\mathcal{P}$ including $\mathcal{P}$ itself is a *subformula* of $\mathcal{P}$. Thus $\sim(A \to B)$ has subformulas,

$$A \qquad B \qquad (A \to B) \qquad \sim(A \to B)$$

In contrast, $(\sim A \to B)$ has subformulas,

$$A \qquad B \qquad \sim A \qquad (\sim A \to B)$$

So it matters for the subformulas how the tree is built. The *immediate* subformulas of a formula $\mathcal{P}$ are the subformulas to which $\mathcal{P}$ is directly connected by lines. Thus $\sim(A \to B)$ has one immediate subformula, $(A \to B)$; $(\sim A \to B)$ has two, $\sim A$ and B. The *atomic* subformulas of a formula $\mathcal{P}$ are the sentence letters that appear across the top row of its tree. Thus both $\sim(A \to B)$ and $(\sim A \to B)$ have A and B as their atomic subformulas. Finally, the *main operator* of a formula $\mathcal{P}$ is the last operator added in its tree. Thus $\sim$ is the main operator of $\sim(A \to B)$, and $\to$ is the main operator of $(\sim A \to B)$. So, again, it matters how the tree is built. We sometimes speak of a formula by means of its main operator: A formula of the form $\sim\mathcal{P}$ is a *negation*; a formula of the form $(\mathcal{P} \to \mathcal{Q})$ is a *(material) conditional*, where $\mathcal{P}$ is the *antecedent* of the conditional and $\mathcal{Q}$ is the *consequent*.

Parts of a Formula

The parts of a formula are here defined in relation to its tree.

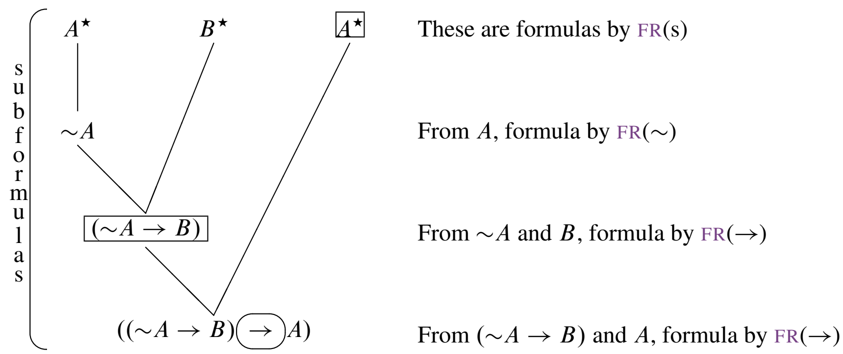
SB Each formula which appears in the tree for formula $\mathcal{P}$ including $\mathcal{P}$ itself is a *subformula* of $\mathcal{P}$.

IS The *immediate* subformulas of a formula $\mathcal{P}$ are the subformulas to which $\mathcal{P}$ is directly connected by lines.

AS The *atomic* subformulas of a formula $\mathcal{P}$ are the sentence letters that appear across the top row of its tree.

MO The *main operator* of a formula $\mathcal{P}$ is the last operator added in its tree.

E2.3. For each of the following expressions, demonstrate that it is a formula and a sentence of $\mathcal{L}_s$ with a tree. Then on the tree (i) bracket all the subformulas, (ii) box the immediate subformula(s), (iii) star the atomic subformulas, and (iv) circle the main operator. A first case for $((\sim A \to B) \to A)$ is worked as an example.



*a. A

b. $\sim\sim\sim A$

c. $\sim(\sim A \to B)$

d. $(\sim C \to \sim(A \to \sim B))$

e. $(\sim(A \to B) \to (C \to \sim A))$

E2.4. Explain why the following expressions are not formulas or sentences of $\mathcal{L}_s$. Hint: you may find that an attempted tree will help you see what is wrong.

a. $(A \supset B)$

*b. $(\mathscr{P} \to \mathscr{Q})$

c. $(\sim B)$

d. $(A \to \sim B \to C)$

e. $((A \to B) \to \sim(A \to C) \to D)$

E2.5. For each of the following expressions, determine whether it is a formula and sentence of $\mathcal{L}_s$. If it is, show it on a tree, and exhibit its parts as in E2.3. If it is not, explain why as in E2.4.

*a. $\sim((A \to B) \to (\sim(A \to B) \to A))$

b. $\sim(A \to B \to (\sim(A \to B) \to A))$

*c. $\sim(A \to B) \to (\sim(A \to B) \to A)$

d. $\sim\sim\sim(\sim\sim\sim A \rightarrow \sim\sim\sim A)$

e. $((\sim(A \rightarrow B) \rightarrow (\sim C \rightarrow D)) \rightarrow \sim(\sim(E \rightarrow F) \rightarrow G))$

2.2.3 Abbreviations

We have completed the official grammar for our sentential languages. So far, the languages are relatively simple. When we turn to reasoning about logic (in later parts), it will be good to have our languages as simple as we can. However, for applications of logic, it will be advantageous to have additional expressions which, though redundant with expressions of the language already introduced, simplify the work. I begin by introducing these additional expressions, and then turn to the question about how to understand the redundancy.

Abbreviating. As may already be obvious, formulas of a sentential language like $\mathcal{L}_s$ can get complicated quickly. Abbreviated forms give us ways to manipulate official expressions without undue pain. First, for any formulas $\mathcal{P}$ and $\mathcal{Q}$,

AB ($\vee$) $(\mathcal{P} \vee \mathcal{Q})$ abbreviates $(\sim\mathcal{P} \rightarrow \mathcal{Q})$

($\wedge$) $(\mathcal{P} \wedge \mathcal{Q})$ abbreviates $\sim(\mathcal{P} \rightarrow \sim\mathcal{Q})$

($\leftrightarrow$) $(\mathcal{P} \leftrightarrow \mathcal{Q})$ abbreviates $\sim((\mathcal{P} \rightarrow \mathcal{Q}) \rightarrow \sim(\mathcal{Q} \rightarrow \mathcal{P}))$

The last of these is easier than it looks; I say something about this below. $\vee$ is *wedge*, $\wedge$ is *caret*, and $\leftrightarrow$ is *double arrow*. An expression of the form $(\mathcal{P} \vee \mathcal{Q})$ is a *disjunction* with $\mathcal{P}$ and $\mathcal{Q}$ as *disjuncts*; it has the standard reading, ($\mathcal{P}$ *or* $\mathcal{Q}$). An expression of the form $(\mathcal{P} \wedge \mathcal{Q})$ is a *conjunction* with $\mathcal{P}$ and $\mathcal{Q}$ as *conjuncts*; it has the standard reading, ($\mathcal{P}$ *and* $\mathcal{Q}$). An expression of the form $(\mathcal{P} \leftrightarrow \mathcal{Q})$ is a *(material) biconditional*; it has the standard reading, ($\mathcal{P}$ *iff* $\mathcal{Q}$).[2] Again, we do not use ordinary English to define our symbols. All the same, this should suggest how the extra operators extend the range of what we are able to say in a natural way.

With the abbreviations, we are in a position to introduce derived clauses for FR. Suppose $\mathcal{P}$ and $\mathcal{Q}$ are formulas; then by FR($\sim$), $\sim\mathcal{P}$ is a formula; so by FR($\rightarrow$), $(\sim\mathcal{P} \rightarrow \mathcal{Q})$ is a formula; but this is just to say that $(\mathcal{P} \vee \mathcal{Q})$ is a formula. And similarly in the other cases. (If you are confused by such reasoning, work it out on a tree.) Thus we arrive at the following conditions.

FR′ ($\vee$) If $\mathcal{P}$ and $\mathcal{Q}$ are formulas, then $(\mathcal{P} \vee \mathcal{Q})$ is a *formula*.

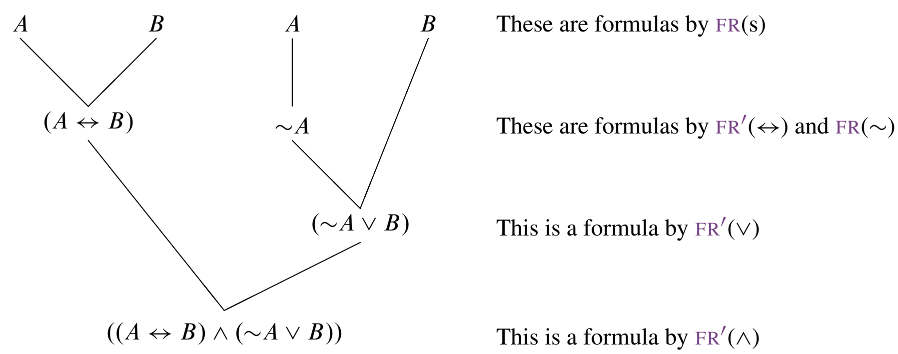
[2] Common alternatives are & for $\wedge$, and $\equiv$ for $\leftrightarrow$.

($\wedge$) If $\mathcal{P}$ and $\mathcal{Q}$ are formulas, then $(\mathcal{P} \wedge \mathcal{Q})$ is a *formula*.

($\leftrightarrow$) If $\mathcal{P}$ and $\mathcal{Q}$ are formulas, then $(\mathcal{P} \leftrightarrow \mathcal{Q})$ is a *formula*.

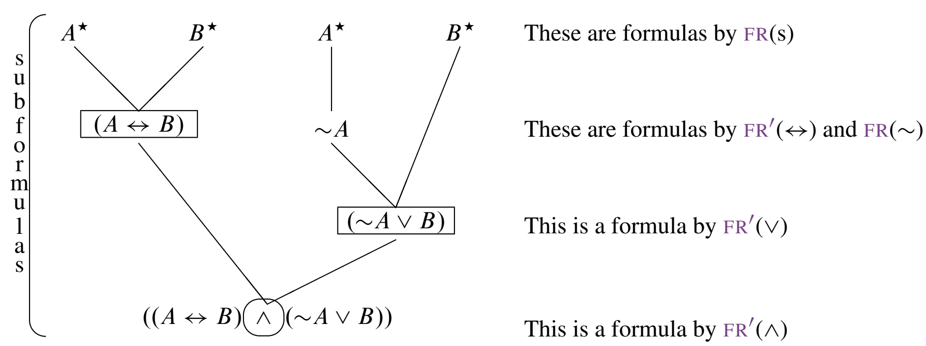
Once FR is extended in this way, the additional conditions may be applied directly in trees. Thus, for example, if $\mathcal{P}$ is a formula and $\mathcal{Q}$ is a formula, we can safely move in a tree to the conclusion that $(\mathcal{P} \vee \mathcal{Q})$ is a formula by FR$'(\vee)$. Similarly, for a more complex case, $((A \leftrightarrow B) \wedge (\sim A \vee B))$ is a formula.

(G)



In a derived sense, expressions with the new symbols have *subformulas*, *atomic* subformulas, *immediate* subformulas, and *main operator* all as before. Thus, with notation from exercises, with bracket for subformulas, star for atomic subformulas, box for immediate subformulas and circle for main operator, on the diagram immediately above,

(H)



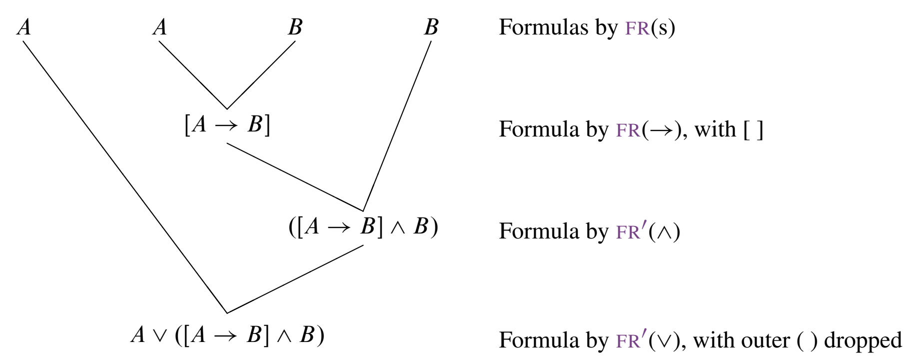
In the derived sense, $((A \leftrightarrow B) \wedge (\sim A \vee B))$ has immediate subformulas $(A \leftrightarrow B)$ and $(\sim A \vee B)$, and main operator $\wedge$.

Return to the case of $(\mathcal{P} \leftrightarrow \mathcal{Q})$ and observe that it can be thought of as based on a simple abbreviation of the sort we expect. That is, $((\mathcal{P} \to \mathcal{Q}) \wedge (\mathcal{Q} \to \mathcal{P}))$ is of the sort $(\mathcal{A} \wedge \mathcal{B})$; so by AB$(\wedge)$, it abbreviates $\sim(\mathcal{A} \to \sim\mathcal{B})$; but with $(\mathcal{P} \to \mathcal{Q})$ for $\mathcal{A}$ and $(\mathcal{Q} \to \mathcal{P})$ for $\mathcal{B}$, this is just, $\sim((\mathcal{P} \to \mathcal{Q}) \to \sim(\mathcal{Q} \to \mathcal{P}))$ as in AB$(\leftrightarrow)$. So you may think of $(\mathcal{P} \leftrightarrow \mathcal{Q})$ as an abbreviation of $((\mathcal{P} \rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \rightarrow \mathcal{P}))$, which in turn abbreviates the more complex $\sim((\mathcal{P} \rightarrow \mathcal{Q}) \rightarrow \sim(\mathcal{Q} \rightarrow \mathcal{P}))$. This is what we expect: a double arrow is like an arrow going from $\mathcal{P}$ to $\mathcal{Q}$ *and* an arrow going from $\mathcal{Q}$ to $\mathcal{P}$.

A couple of additional abbreviations concern parentheses. First, it is sometimes convenient to use a pair of square brackets [ ] in place of parentheses ( ). This is purely for visual convenience; for example ((()())) may be more difficult to absorb than ([()()]). Second, if the very last step of a tree for some formula $\mathcal{P}$ is justified by FR($\rightarrow$), FR$'$($\wedge$), FR$'$($\vee$), or FR$'$($\leftrightarrow$), we feel free to abbreviate $\mathcal{P}$ with the *outermost* set of parentheses or brackets dropped. Again, this is purely for visual convenience. Thus, for example, we might write, $A \rightarrow (B \rightarrow C)$ in place of $(A \rightarrow (B \rightarrow C))$. As it turns out, where $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are formulas, there is a difference between $((\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{C})$ and $(\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C}))$, insofar as the main operator shifts from one case to the other. In $(\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C})$, however, it is not clear which arrow should be the main operator. That is why we do not count the latter as a grammatical formula or sentence. Similarly there is a difference between $\sim(\mathcal{A} \rightarrow \mathcal{B})$ and $(\sim\mathcal{A} \rightarrow \mathcal{B})$; again, the main operator shifts. However, there is no room for ambiguity when we drop just an outermost pair of parentheses and write $(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{C}$ for $((\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{C})$; and similarly when we write $\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})$ for $(\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C}))$. The same reasoning applies for abbreviations with $\wedge$, $\vee$, or $\leftrightarrow$. So dropping outermost parentheses counts as a legitimate abbreviation.

An expression which uses the extra operators, square brackets, or drops outermost parentheses is a formula just insofar as it is a sort of shorthand for an official formula which does not. But we will not usually distinguish between the shorthand expressions and official formulas. Thus, again, the new conditions may be applied directly in trees and, for example, the following is a legitimate tree to demonstrate that $A \vee ([A \rightarrow B] \wedge B)$ is a formula.

(I)

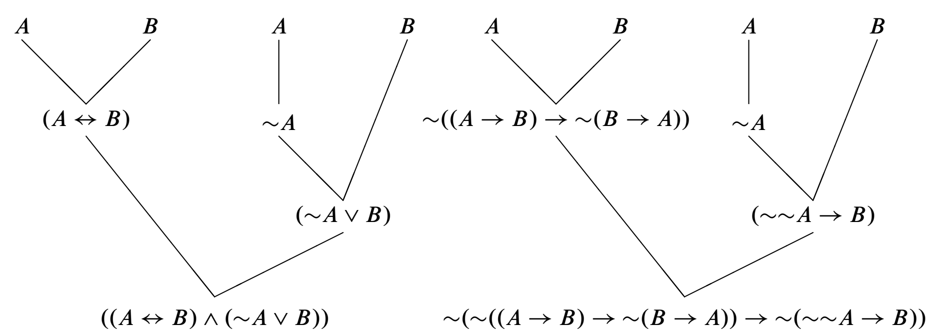


So we use our extra conditions for FR$'$, introduce square brackets instead of parentheses, and drop parentheses in the very last step. Remember that the *only* case where you can omit parentheses is if they would have been added in the very last step of the tree. So long as we do not distinguish between shorthand expressions and official formulas, we regard a tree of this sort as sufficient to demonstrate that an expression is a formula and a sentence.

Unabbreviating. As we have suggested, there is a certain tension between the advantages of a simple language, and one that is more complex. When a language is simple, it is easier to reason about; when it has additional resources, it is easier to use. Expressions with $\wedge$, $\vee$ and $\leftrightarrow$ are redundant with expressions that do not have them — though it is easier to work with a language that has $\wedge$, $\vee$ and $\leftrightarrow$ than with one that does not (something like reciting the Pledge of Allegiance in English, and then in Morse code; you can do it in either, but it is easier in the former). If all we wanted was a simple language to reason about, we would forget about the extra operators. If all we wanted was a language easy to use, we would forget about keeping the language simple. To have the advantages of both, we have adopted the position that expressions with the extra operators *abbreviate*, or are a shorthand for, expressions of the original language. It will be convenient to work with abbreviations in many contexts. But, when it comes to reasoning about the language, we set the abbreviations to the side, and focus on the official language itself.

For this to work, we have to be able to undo abbreviations when required. It is, of course, easy enough to substitute parentheses back for square brackets, or to replace outermost dropped parentheses. For formulas with the extra operators, it is always possible to work through trees, using AB to replace formulas with unabbreviated forms, one operator at a time. Consider an example.

(J)

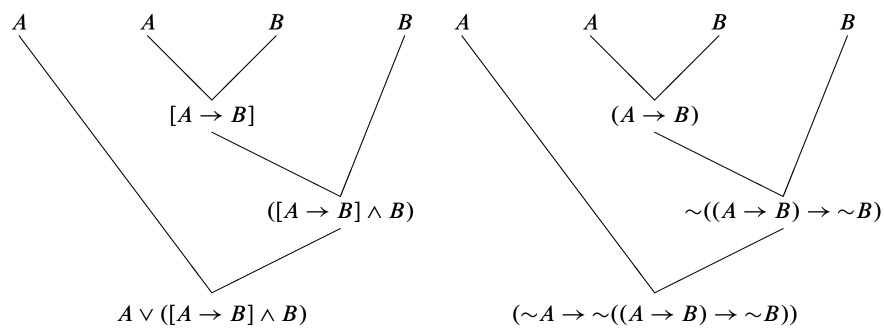


The tree on the left is (G) from above. The tree on the right uses AB to "unpack" each of the expressions on the left. Atomics remain as before. Then, at each stage, given an unabbreviated version of the parts, we give an unabbreviated version of the whole. First, $(A \leftrightarrow B)$ abbreviates $\sim((A \to B) \to \sim(B \to A))$; this is a simple application of AB($\leftrightarrow$). $\sim A$ is not an abbreviation and so remains as before. From AB($\vee$), $(\mathcal{P} \vee \mathcal{Q})$

abbreviates $(\sim\mathcal{P} \rightarrow \mathcal{Q})$ so $(\sim A \vee B)$ abbreviates tilde the left disjunct, arrow the right (so that we get two tildes). For the final result, we combine the input formulas according to the unabbreviated form for $\wedge$. It is more a bookkeeping problem than anything: There is one formula $\mathcal{P}$ that is $(A \leftrightarrow B)$, another $\mathcal{Q}$ that is $(\sim A \vee B)$; these are combined into $(\mathcal{P} \wedge \mathcal{Q})$ and so, by AB($\wedge$), into $\sim(\mathcal{P} \rightarrow \sim\mathcal{Q})$. You should be able to see that this is just what we have done. There is a tilde and a parenthesis; then the $\mathcal{P}$; then an arrow and a tilde; then the $\mathcal{Q}$, and a closing parenthesis. Not only is the abbreviation more compact but, as we shall see, there is a corresponding advantage when it comes to grasping what an expression says.

Here is a another example, this time from (I). In this case, we replace also square brackets and restore dropped outer parentheses.

(K)



In the right hand tree, we reintroduce parentheses for the square brackets. Similarly, we apply AB($\wedge$) and AB($\vee$) to unpack shorthand symbols. And outer parentheses are reintroduced at the very last step. Thus $A \vee ([A \rightarrow B] \wedge B)$ is a shorthand for the unabbreviated expression, $(\sim A \rightarrow \sim((A \rightarrow B) \rightarrow \sim B))$.

Observe that right-hand trees are *not* ones of the sort you would use directly to show that an expression is a formula by FR! FR does not let you move directly from that $(A \rightarrow B)$ is a formula and B is a formula, to the result that $\sim((A \rightarrow B) \rightarrow \sim B)$ is a formula as just above. Of course, if $(A \rightarrow B)$ and B are formulas, then $\sim((A \rightarrow B) \rightarrow \sim B)$ is a formula, and nothing stops a tree to show it. This is the point of our derived clauses for FR$'$. In fact, this is a good check on your unabbreviations: If the result is not a formula, you have made a mistake! But you should not think of trees as on the right as involving application of FR. Rather they are *unabbreviating* trees, with application of AB to shorthand expressions from trees as on the left. A fully unabbreviated expression always meets all the requirements from section 2.2.2.

E2.6. For each of the following expressions, demonstrate that it is a formula and a sentence of $\mathcal{L}_s$ with a tree. Then on the tree (i) bracket all the subformulas,

(ii) box the immediate subformula(s), (iii) star the atomic subformulas, and (iv) circle the main operator.

*a. $(A \wedge B) \rightarrow C$

b. $\sim([A \rightarrow \sim K_{14}] \vee C_3)$

c. $B \rightarrow (\sim A \leftrightarrow B)$

d. $(B \rightarrow A) \wedge (C \vee A)$

e. $(A \vee \sim B) \leftrightarrow (C \wedge A)$

*E2.7. For each of the formulas in E2.6a - e, produce an unabbreviating tree to find the unabbreviated expression it represents.

*E2.8. For each of the unabbreviated expressions from E2.7a - e, produce a complete tree to show by direct application of FR that it is an official formula.

E2.9. In the text, we introduced derived clauses to FR by reasoning as follows, "Suppose $\mathcal{P}$ and $\mathcal{Q}$ are formulas; then by FR($\sim$), $\sim\mathcal{P}$ is a formula; so by FR($\rightarrow$), $(\sim\mathcal{P} \rightarrow \mathcal{Q})$ is a formula; but this is just to say that $(\mathcal{P} \vee \mathcal{Q})$ is a formula. And similarly in the other cases" (p. 42). Supposing that $\mathcal{P}$ and $\mathcal{Q}$ are formulas, produce the similar reasoning to show that $(\mathcal{P} \wedge \mathcal{Q})$ and $(\mathcal{P} \leftrightarrow \mathcal{Q})$ are formulas. Hint: Again, it may help to think about trees.

E2.10. For each of the following concepts, explain in an essay of about two pages, so that (high-school age) Hannah could understand. In your essay, you should (i) identify the objects to which the concept applies, (ii) give and explain the definition, and give and explicate examples of your own construction (iii) where the concept applies, and (iv) where it does not. Your essay should exhibit an understanding of methods from the text.

a. The vocabulary for a sentential language, and use of the metalanguage.

b. A formula of a sentential language.

c. The parts of a formula.

d. The abbreviation and unabbreviation for an official formula of a sentential language.

2.3 Quantificational Languages

Chapter 3

Axiomatic Deduction

We have not yet said what our sentences mean. This is just what we do in the next chapter. However, just as it is possible to do grammar without reference to meaning, so it is possible to do derivations without reference to meaning. Derivations are *defined* purely in relation to formula and sentence form. That is why it is crucial to *show* that derivations stand in important relations to validity and truth, as we do in **??**. And that is why it is possible to do derivations without knowing what the expressions mean. In this chapter we develop an *axiomatic* derivation system without any reference to meaning and truth. Apart from relations to meaning and truth, derivations are perfectly well-defined — counting at least as a sort of puzzle or game with, perhaps, a related "thrill of victory" and "agony of defeat." And as with a game, it is possible to build derivation skills, to become a better player. Later, we will show how derivation games matter.[1]

Derivation systems are constructed for different purposes. Introductions to mathematical logic typically employ an *axiomatic* approach. We will see a *natural deduction* system in chapter 6. The advantage of axiomatic systems is their extreme simplicity. From a practical point of view, when we want to think *about* logic, it is convenient to have a relatively simple object to think about. The axiomatic approach makes it natural to build toward increasingly complex and powerful results. As we will see, however, in the beginning, axiomatic derivations can be relatively challenging! We will introduce our system in stages: After some general remarks about what an axiom system is supposed to be, we will introduce the sentential component of our system —

[1]This chapter is out of place. Having developed the grammar of our formal languages, a sensible course in mathematical logic will skip directly to chapter 4 and return only after chapter 6. This chapter has its location to crystallize the the point about form. One might reasonably attempt the first section, but then return only after background from chapters that follow.

the part with application to forms involving just $\sim$ and $\to$ (and so $\vee$, $\wedge$, and $\leftrightarrow$). After that, we will turn to the full system for forms with quantifiers and equality, including a mathematical application.

3.1 General

Before turning to the derivations themselves, it will be helpful to make some points about the metalanguage and form. First, we are familiar with the idea that different formulas may be of the same form. Thus, for example, where $\mathcal{P}$ and $\mathcal{Q}$ are formulas, $A \to B$ and $A \to (B \vee C)$ are both of the form, $\mathcal{P} \to \mathcal{Q}$ — in the one case $\mathcal{Q}$ maps to B, and in the other to $(B \vee C)$. And, more generally, for formulas $\mathcal{A}, \mathcal{B}, \mathcal{C}$, any formula of the form $\mathcal{A} \to (\mathcal{B} \vee \mathcal{C})$ is also of the form $\mathcal{P} \to \mathcal{Q}$. For if $(\mathcal{B} \vee \mathcal{C})$ maps onto some formula, $\mathcal{Q}$ maps onto that formula as well. Of course, this does not go the other way around: it is not the case that every expression of the form $\mathcal{P} \to \mathcal{Q}$ is of the form $\mathcal{A} \to (\mathcal{B} \vee \mathcal{C})$; for it is not the case that $\mathcal{B} \vee \mathcal{C}$ maps to any expression to onto which $\mathcal{Q}$ maps. Be sure you are clear about this! Using the metalanguage this way, we can speak generally about formulas in arbitrary sentential or quantificational languages. This is just what we will do — on the assumption that our script letters $\mathcal{A} \ldots \mathcal{Z}$ range over formulas of some arbitrary formal language $\mathcal{L}$, we frequently depend on the fact that every formula of one form is also of another.

Given a formal language $\mathcal{L}$, an axiomatic logic *AL* consists of two parts. There is a set of *axioms* and a set of *rules*. Different axiomatic logics result from different axioms and rules. For now, the set of axioms is just some privileged collection of formulas. A rule tells us that one formula *follows* from some others. One way to specify axioms and rules is by form. Thus, for example, *modus ponens* may be included among the rules.

MP $$\frac{\mathcal{P} \to \mathcal{Q},\ \mathcal{P}}{\mathcal{P}}$$

According to this rule, for any formulas $\mathcal{P}$ and $\mathcal{Q}$, the formula $\mathcal{Q}$ *follows* from $\mathcal{P} \to \mathcal{Q}$ together with $\mathcal{P}$. Thus, as applied to $\mathcal{L}_s$, B follows by MP from $A \to B$ and A; but also $(B \leftrightarrow D)$ follows from $(A \to B) \to (B \leftrightarrow D)$ and $(A \to B)$. And for a case put in the metalanguage, quite generally, a formula of the form $(\mathcal{A} \wedge \mathcal{B})$ follows from $\mathcal{A} \to (\mathcal{A} \wedge \mathcal{B})$ and $\mathcal{A}$ — for any formulas of the form $\mathcal{A} \to (\mathcal{A} \wedge \mathcal{B})$ and $\mathcal{A}$ are of the forms $\mathcal{P} \to \mathcal{Q}$ and $\mathcal{P}$ as well. Axioms also may be specified by form. Thus, for some language with formulas $\mathcal{P}$ and $\mathcal{Q}$, a logic might include all formulas of the forms,

$\wedge 1$ $(\mathcal{P} \wedge \mathcal{Q}) \to \mathcal{P}$ $\wedge 2$ $(\mathcal{P} \wedge \mathcal{Q}) \to \mathcal{Q}$ $\wedge 3$ $\mathcal{P} \to (\mathcal{Q} \to (\mathcal{P} \wedge \mathcal{Q}))$

among its axioms. Then in $\mathcal{L}_s$,

$$(A \wedge B) \to A, \qquad (A \wedge A) \to A \qquad ((A \to B) \wedge C) \to (A \to B)$$

are all axioms of form ∧1. So far, for a given axiomatic logic *AL*, there are no constraints on just which forms will be the axioms, and just which rules are included. The point is only that we specify an axiomatic logic when we specify some collection of axioms and rules.

Suppose we have specified some axioms and rules for an axiomatic logic *AL*. Then where Γ (Gamma), is a set of formulas — taken as the formal *premises* of an argument,

AV (p) If $\mathcal{P}$ is a premise (a member of Γ), then $\mathcal{P}$ is a *consequence* in *AL* of Γ.

(a) If $\mathcal{P}$ is an axiom of *AL*, then $\mathcal{P}$ is a *consequence* in *AL* of Γ.

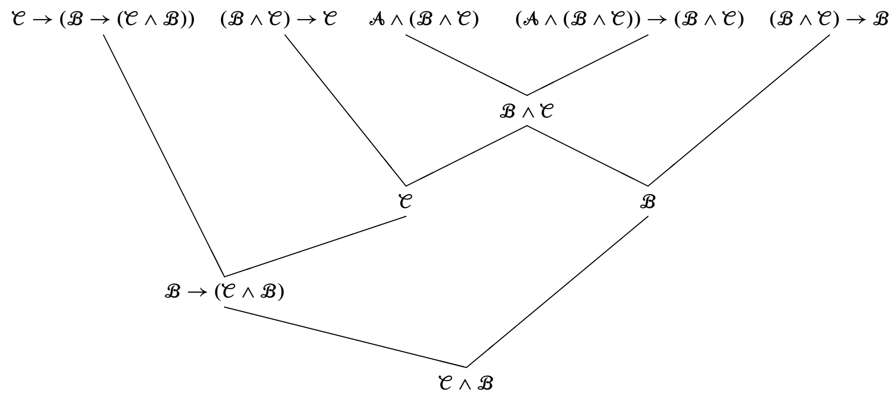
(r) If $\mathcal{Q}_1 \dots \mathcal{Q}_n$ are consequences in *AL* of Γ, and there is a rule of *AL* such that $\mathcal{P}$ follows from $\mathcal{Q}_1 \dots \mathcal{Q}_n$ by the rule, then $\mathcal{P}$ is a *consequence* in *AL* of Γ.

(CL) Any *consequence* in *AL* of Γ may be obtained by repeated application of these rules.

The first two clauses make premises and axioms consequences in *AL* of Γ. And if, say, MP is a rule of an *AL* and $P \to Q$ and P are consequences in *AL* of Γ, then by AV(r), Q is a consequence in *AL* of Γ as well. If $\mathcal{P}$ is a consequence in *AL* of some premises Γ, then the premises *prove* $\mathcal{P}$ in *AL* and equivalently the argument is *valid* in *AL*; in this case we write $\Gamma \vdash_{AL} \mathcal{P}$. The ⊢ symbol is the *single turnstile* (to contrast with a *double* turnstile ⊨ from chapter 4). If $\mathcal{Q}_1 \dots \mathcal{Q}_n$ are the members of Γ, we sometimes write $\mathcal{Q}_1 \dots \mathcal{Q}_n \vdash_{AL} \mathcal{P}$ in place of $\Gamma \vdash_{AL} \mathcal{P}$. If Γ has no members at all and $\Gamma \vdash_{AL} \mathcal{P}$, then $\mathcal{P}$ is a *theorem* of *AL*. In this case, listing all the premises individually, we simply write, $\vdash_{AL} \mathcal{P}$.

Before turning to our official axiomatic system *AD*, it will be helpful to consider a simple example. Suppose an axiomatic derivation system *A1* has MP as its only rule, and just formulas of the forms ∧1, ∧2, and ∧3 as axioms. AV is a recursive definition like ones we have seen before. Thus nothing stops us from working out its consequences on trees. Thus we can show that $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C}) \vdash_{A1} \mathcal{C} \wedge \mathcal{B}$ as follows,

(A)



For definition AV, the basic elements are the premises and axioms. These occur across the top row. Thus, reading from the left, the first form is an instance of $\land$3. The second is of type $\land$2. These are thus consequences of Γ by AV(a). The third is the premise. Thus it is a consequence by AV(p). Any formula of the form $(\mathcal{A} \land (\mathcal{B} \land \mathcal{C})) \to (\mathcal{B} \land \mathcal{C})$ is of the form, $(\mathcal{P} \land \mathcal{Q}) \to \mathcal{Q}$; so the fourth is of the type $\land$2. And the last is of the type $\land$1. So the final two are consequences by AV(a). After that, all the results are by MP, and so consequences by AV(r). Thus for example, in the first case, $(\mathcal{A} \land (\mathcal{B} \land \mathcal{C})) \to (\mathcal{B} \land \mathcal{C})$ and $\mathcal{A} \land (\mathcal{B} \land \mathcal{C})$ are of the sort $\mathcal{P} \to \mathcal{Q}$ and $\mathcal{P}$, with $\mathcal{A} \land (\mathcal{B} \land \mathcal{C})$ for $\mathcal{P}$ and $(\mathcal{B} \land \mathcal{C})$ for $\mathcal{Q}$; thus $\mathcal{B} \land \mathcal{C}$ follows from them by MP. So $\mathcal{B} \land \mathcal{C}$ is a consequence in *A1* of Γ by AV(r). And similarly for the other consequences. Notice that applications of MP and of the axiom forms are *independent* from one use to the next. The expressions that count as $\mathcal{P}$ or $\mathcal{Q}$ must be consistent within a given application of the axiom or rule, but may vary from one application of the axiom or rule to the next. If you are familiar with another derivation system, perhaps the one from chapter 6, you may think of an axiom as a rule without inputs. Then the axiom applies to expressions of its form in the usual way.

These diagrams can get messy, and it is traditional to represent the same information as follows, using annotations to indicate relations among formulas.

(B)

1.	$\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$	p(remise)
2.	$(\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})) \rightarrow (\mathcal{B} \wedge \mathcal{C})$	$\wedge$2
3.	$\mathcal{B} \wedge \mathcal{C}$	2,1 MP
4.	$(\mathcal{B} \wedge \mathcal{C}) \rightarrow \mathcal{B}$	$\wedge$1
5.	$\mathcal{B}$	4,3 MP
6.	$(\mathcal{B} \wedge \mathcal{C}) \rightarrow \mathcal{C}$	$\wedge$2
7.	$\mathcal{C}$	6,3 MP
8.	$\mathcal{C} \rightarrow (\mathcal{B} \rightarrow (\mathcal{C} \wedge \mathcal{B}))$	$\wedge$3
9.	$\mathcal{B} \rightarrow (\mathcal{C} \wedge \mathcal{B})$	8,7 MP
10.	$\mathcal{C} \wedge \mathcal{B}$	9,5 MP

Each of the forms (1) - (10) is a consequence of $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$ in *A1*. As indicated on the right, the first is a premise, and so a consequence by AV(p). The second is an axiom of the form $\wedge$2, and so a consequence by AV(a). The third follows by MP from the forms on lines (2) and (1), and so is a consequence by AV(r). And so forth. Such a demonstration is an *axiomatic derivation*. This derivation contains the very same information as the tree diagram (A), only with geometric arrangement replaced by line numbers to indicate relations between forms. Observe that we might have accomplished the same end with a different arrangement of lines. For example, we might have listed all the axioms first, with applications of MP after. The important point is that in an *axiomatic derivation*, each line is either an axiom, a premise, or follows from previous lines by a rule. Just as a tree is sufficient to demonstrate that $\Gamma \vdash_{AL} \mathcal{P}$, that $\mathcal{P}$ is a consequence of Γ in *AL*, so an axiomatic derivation is sufficient to show the same. In fact, we shall typically use derivations, rather than trees to show that $\Gamma \vdash_{AL} \mathcal{P}$.

Notice that we have been reasoning with sentence *forms*, and so have shown that a formula of the form $\mathcal{C} \wedge \mathcal{B}$ follows in *A1* from one of the form $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$. Given this, we freely appeal to results of one derivation in the process of doing another. Thus, if we were to encounter a formula of the form $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$ in an *A1* derivation, we might simply cite the derivation (B) completed above, and move directly to the conclusion that $\mathcal{C} \wedge \mathcal{B}$. The resultant derivation would be an *abbreviation* of an official one which includes each of the above steps to reach $\mathcal{C} \wedge \mathcal{B}$. In this way, derivations remain manageable, and we are able to build toward results of increasing complexity. (Compare your high school experience of Euclidian geometry.) All of this should become more clear, as we turn to the official and complete axiomatic system, *AD*.

E3.1. Where *A1* is as above with rule MP and axioms $\wedge$1-3, construct derivations to show each of the following.

*a. $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C}) \vdash_{A1} \mathcal{B}$

b. $\mathcal{A}, \mathcal{B}, \mathcal{C} \vdash_{A1} \mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$

c. $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C}) \vdash_{A1} (\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C}$

d. $(\mathcal{A} \wedge \mathcal{B}) \wedge (\mathcal{C} \wedge \mathcal{D}) \vdash_{A1} \mathcal{B} \wedge \mathcal{C}$

e. $\vdash_{A1} ((\mathcal{A} \wedge \mathcal{B}) \to \mathcal{A}) \wedge ((\mathcal{A} \wedge \mathcal{B}) \to \mathcal{B})$

Again, unless you have a special reason for studying axiomatic systems, or are just looking for some really challenging puzzles, at this stage you should move on to the next chapter and return only after chapter 6. This chapter makes sense here to underline the conceptual point that derivations are defined apart from notions of validity and truth, but it is completely out of order from a learning point of view. After chapter 6 you can return to this chapter, but recognize its place in the conceptual order.

3.2 Sentential

We begin by focusing on sentential forms, forms involving just $\sim$ and $\to$ (and so $\wedge$, $\vee$ and $\leftrightarrow$). The sentential component of our official axiomatic logic *AD* tells us how to manipulate such forms, whether they be forms for expressions in a sentential language like $\mathcal{L}_s$, or in a quantificational language like $\mathcal{L}_q$. The sentential fragment of *AD* includes three forms for logical axioms, and one rule.

AS A1. $\mathcal{P} \to (\mathcal{Q} \to \mathcal{P})$

A2. $(\mathcal{O} \to (\mathcal{P} \to \mathcal{Q})) \to ((\mathcal{O} \to \mathcal{P}) \to (\mathcal{O} \to \mathcal{Q}))$

A3. $(\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})$

MP $\mathcal{Q}$ follows from $\mathcal{P} \to \mathcal{Q}$ and $\mathcal{P}$

We have already encountered MP. To take some cases to appear immediately below, the following are both of the sort A1.

$$\mathcal{A} \to (\mathcal{A} \to \mathcal{A}) \qquad (\mathcal{B} \to \mathcal{C}) \to [\mathcal{A} \to (\mathcal{B} \to \mathcal{C})]$$

Observe that $\mathcal{P}$ and $\mathcal{Q}$ need not be different! You should be clear about these cases. Although MP is the only rule, we allow free movement between an expression and its abbreviated forms, with justification, "abv." That is it! As above, $\Gamma \vdash_{ADs} \mathcal{P}$ just in case $\mathcal{P}$ is a consequence of Γ in *AD*. $\Gamma \vdash_{ADs} \mathcal{P}$ just in case there is a derivation of $\mathcal{P}$ from premises in Γ.

The following is a series of derivations where, as we shall see, each may depend on ones from before. At first, do not worry so much about strategy, as about the mechanics of the system.

T3.1. $\vdash_{ADs} \mathcal{A} \to \mathcal{A}$

1.	$\mathcal{A} \to ([\mathcal{A} \to \mathcal{A}] \to \mathcal{A})$	A1
2.	$(\mathcal{A} \to ([\mathcal{A} \to \mathcal{A}] \to \mathcal{A})) \to ((\mathcal{A} \to [\mathcal{A} \to \mathcal{A}]) \to (\mathcal{A} \to \mathcal{A}))$	A2
3.	$(\mathcal{A} \to [\mathcal{A} \to \mathcal{A}]) \to (\mathcal{A} \to \mathcal{A})$	2,1 MP
4.	$\mathcal{A} \to [\mathcal{A} \to \mathcal{A}]$	A1
5.	$\mathcal{A} \to \mathcal{A}$	3,4 MP

Line (1) is an axiom of the form A1 with $\mathcal{A} \to \mathcal{A}$ for $\mathcal{Q}$. Line (2) is an axiom of the form A2 with $\mathcal{A}$ for $\mathcal{O}$, $\mathcal{A} \to \mathcal{A}$ for $\mathcal{P}$, and $\mathcal{A}$ for $\mathcal{Q}$. Notice again that $\mathcal{O}$ and $\mathcal{Q}$ may be any formulas, so nothing prevents them from being the same. Similarly, line (4) is an axiom of form A1 with $\mathcal{A}$ in place of both $\mathcal{P}$ and $\mathcal{Q}$. The applications of MP should be straightforward.

T3.2. $\mathcal{A} \to \mathcal{B}, \mathcal{B} \to \mathcal{C} \vdash_{ADs} \mathcal{A} \to \mathcal{C}$

1.	$\mathcal{B} \to \mathcal{C}$	prem
2.	$(\mathcal{B} \to \mathcal{C}) \to [\mathcal{A} \to (\mathcal{B} \to \mathcal{C})]$	A1
3.	$\mathcal{A} \to (\mathcal{B} \to \mathcal{C})$	2,1 MP
4.	$[\mathcal{A} \to (\mathcal{B} \to \mathcal{C})] \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})]$	A2
5.	$(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})$	4,3 MP
6.	$\mathcal{A} \to \mathcal{B}$	prem
7.	$\mathcal{A} \to \mathcal{C}$	5,6 MP

Line (4) is an instance of A2 which gives us our goal with two applications of MP — that is, from (4), $\mathcal{A} \to \mathcal{C}$ follows by MP if we have $\mathcal{A} \to (\mathcal{B} \to \mathcal{C})$ and $\mathcal{A} \to \mathcal{B}$. But the second of these is a premise, so the only real challenge is getting $\mathcal{A} \to (\mathcal{B} \to \mathcal{C})$. But since $\mathcal{B} \to \mathcal{C}$ is a premise, we can use A1 to get *anything* arrow it — and that is just what we do by the first three lines.

T3.3. $\mathcal{A} \to (\mathcal{B} \to \mathcal{C}) \vdash_{ADs} \mathcal{B} \to (\mathcal{A} \to \mathcal{C})$

1.	$\mathcal{B} \to (\mathcal{A} \to \mathcal{B})$	A1
2.	$\mathcal{A} \to (\mathcal{B} \to \mathcal{C})$	prem
3.	$[\mathcal{A} \to (\mathcal{B} \to \mathcal{C})] \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})]$	A2
4.	$(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})$	3,2 MP
5.	$\mathcal{B} \to (\mathcal{A} \to \mathcal{C})$	1,4 T3.2

In this case, the first four steps are very much like ones you have seen before. But the last is not. We have $\mathcal{B} \to (\mathcal{A} \to \mathcal{B})$ on line (1), and $(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})$ on line (4). These are of the form to be inputs to T3.2 — with $\mathcal{B}$ for $\mathcal{A}$, $\mathcal{A} \to \mathcal{B}$ for $\mathcal{B}$, and $\mathcal{A} \to \mathcal{C}$ for $\mathcal{C}$. T3.2 is a sort of transitivity or "chain" principle which lets us move from a first form to a last through some middle term. In this case, $\mathcal{A} \to \mathcal{B}$ is the middle term. So at line (5), we simply observe that lines (1) and (4), together with the reasoning from T3.2, give us the desired result.

What we have not produced is an official derivation, where each step is a premise, an axiom, or follows from previous lines by a rule. But we have produced an abbreviation of one. And nothing prevents us from unabbreviating by including the routine from T3.2 to produce a derivation in the official form. To see this, first, observe that the derivation for T3.2 has its premises at lines (1) and (6), where lines with the corresponding forms in the derivation for T3.3 appear at (4) and (1). However, it is a simple matter to reorder the derivation for T3.2 so that it takes its premises from those same lines. Thus here is another demonstration for T3.2.

(C)

1.	$\mathcal{A} \to \mathcal{B}$	prem
$\vdots$		
4.	$\mathcal{B} \to \mathcal{C}$	prem
5.	$(\mathcal{B} \to \mathcal{C}) \to [\mathcal{A} \to (\mathcal{B} \to \mathcal{C})]$	A1
6.	$\mathcal{A} \to (\mathcal{B} \to \mathcal{C})$	5,4 MP
7.	$[\mathcal{A} \to (\mathcal{B} \to \mathcal{C})] \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})]$	A2
8.	$(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})$	7,6 MP
9.	$\mathcal{A} \to \mathcal{C}$	8,1 MP

Compared to the original derivation for T3.2, all that is different is the order of a few lines, and corresponding line numbers. The *reason* for reordering the lines is for a merge of this derivation with the one for T3.3.

But now, although we are after expressions of the *form* $\mathcal{A} \to \mathcal{B}$ and $\mathcal{B} \to \mathcal{C}$, the actual expressions we want for T3.3 are $\mathcal{B} \to (\mathcal{A} \to \mathcal{B})$ and $(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})$. But we can convert derivation (C) to one with those very forms by uniform substituation of $\mathcal{B}$ for every $\mathcal{A}$; $(\mathcal{A} \to \mathcal{B})$ for every $\mathcal{B}$; and $(\mathcal{A} \to \mathcal{C})$ for every $\mathcal{C}$ — that is, we apply our original map to the entire derivation (C). The result is as follows.

	1.	$\mathcal{B} \to (\mathcal{A} \to \mathcal{B})$	prem
	$\vdots$		
	4.	$(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})$	prem
(D)	5.	$((\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})) \to [\mathcal{B} \to ((\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C}))]$	A1
	6.	$\mathcal{B} \to ((\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C}))$	5,4 MP
	7.	$[\mathcal{B} \to ((\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C}))] \to [(\mathcal{B} \to (\mathcal{A} \to \mathcal{B})) \to (\mathcal{B} \to (\mathcal{A} \to \mathcal{C}))]$	A2
	8.	$(\mathcal{B} \to (\mathcal{A} \to \mathcal{B})) \to (\mathcal{B} \to (\mathcal{A} \to \mathcal{C}))$	7,6 MP
	9.	$\mathcal{B} \to (\mathcal{A} \to \mathcal{C})$	8,1 MP

You should trace the parallel between derivations (C) and (D) all the way through. And you should verify that (D) is a derivation on its own. This is an application of the point that our derivation for T3.2 applies to any premises and conclusions of that form. The result is a direct demonstration that $\mathcal{B} \to (\mathcal{A} \to \mathcal{B}), (\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C}) \vdash_{ADs} \mathcal{B} \to (\mathcal{A} \to \mathcal{C})$.

And now it is a simple matter to merge the lines from (D) into the derivation for T3.3 to produce a complete demonstration that $\mathcal{A} \to (\mathcal{B} \to \mathcal{C}) \vdash_{ADs} \mathcal{B} \to (\mathcal{A} \to \mathcal{C})$.

	1.	$\mathcal{B} \to (\mathcal{A} \to \mathcal{B})$	A1
	2.	$\mathcal{A} \to (\mathcal{B} \to \mathcal{C})$	prem
	3.	$[\mathcal{A} \to (\mathcal{B} \to \mathcal{C})] \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})]$	A2
	4.	$(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})$	3,2 MP
(E)	5.	$((\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})) \to [\mathcal{B} \to ((\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C}))]$	A1
	6.	$\mathcal{B} \to ((\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C}))$	5,4 MP
	7.	$[\mathcal{B} \to ((\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C}))] \to [(\mathcal{B} \to (\mathcal{A} \to \mathcal{B})) \to (\mathcal{B} \to (\mathcal{A} \to \mathcal{C}))]$	A2
	8.	$(\mathcal{B} \to (\mathcal{A} \to \mathcal{B})) \to (\mathcal{B} \to (\mathcal{A} \to \mathcal{C}))$	7,6 MP
	9.	$\mathcal{B} \to (\mathcal{A} \to \mathcal{C})$	8,1 MP

Lines (1) - (4) are the same as from the derivation for T3.3, and include what are the premises to (D). Lines (5) - (9) are the same as from (D). The result is a demonstration for T3.3 in which every line is a premise, an axiom, or follows from previous lines by MP. Again, you should follow each step. It is hard to believe that we could *think up* this last derivation — particularly at this early stage of our career. However, if we can produce the simpler derivation, we can be sure that this more complex one exists. Thus we can be sure that the final result is a consequence of the premise in *AD*. That is the point of our direct appeal to T3.2 in the original derivation of T3.3. And similarly in cases that follow. In general, we are always free to appeal to prior results in any derivation — so that our toolbox gets bigger at every stage.

T3.4. $\vdash_{ADs} (\mathcal{B} \to \mathcal{C}) \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})]$

1. $(\mathcal{B} \rightarrow \mathcal{C}) \rightarrow [\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})]$ A1
2. $[\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})] \rightarrow [(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow \mathcal{C})]$ A2
3. $(\mathcal{B} \rightarrow \mathcal{C}) \rightarrow [(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow \mathcal{C})]$ 1,2 T3.2

Again, we have an application of T3.2. In this case, the middle term (the $\mathcal{B}$) from T3.2 maps to $\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})$. Once we see that the consequent of what we want is like the consequent of A2, we should be "inspired" by T3.2 to go for (1) as a link between the antecedent of what we want, and antecedent of A2. As it turns out, this is easy to get as an instance of A1. It is helpful to say to yourself in words, what the various axioms and theorems do. Thus, given some $\mathcal{P}$, A1 yields *anything* arrow it. And T3.2 is a simple transitivity principle.

T3.5. $\vdash_{ADs} (\mathcal{A} \rightarrow \mathcal{B}) \rightarrow [(\mathcal{B} \rightarrow \mathcal{C}) \rightarrow (\mathcal{A} \rightarrow \mathcal{C})]$

1. $(\mathcal{B} \rightarrow \mathcal{C}) \rightarrow [(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\mathcal{A} \rightarrow \mathcal{C})]$ T3.4
2. $(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow [(\mathcal{B} \rightarrow \mathcal{C}) \rightarrow (\mathcal{A} \rightarrow \mathcal{C})]$ 1 T3.3

T3.5 is like T3.4 except that $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{B} \rightarrow \mathcal{C}$ switch places. But T3.3 precisely switches terms in those places — with $\mathcal{B} \rightarrow \mathcal{C}$ for $\mathcal{A}$, $\mathcal{A} \rightarrow \mathcal{B}$ for $\mathcal{B}$, and $\mathcal{A} \rightarrow \mathcal{C}$ for $\mathcal{C}$. Again, often what is difficult about these derivations is "seeing" what you can do. Thus it is good to say to yourself in words what the different principles give you. Once you realize what T3.3 does, it is obvious that you have T3.5 immediately from T3.4.

T3.6. $\mathcal{B}, \mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C}) \vdash_{ADs} \mathcal{A} \rightarrow \mathcal{C}$

Hint: You can get this in the basic system using just A1 and A2. But you can get it in just four lines if you use T3.3.

T3.7. $\vdash_{ADs} (\sim\mathcal{A} \rightarrow \mathcal{A}) \rightarrow \mathcal{A}$

Hint: This follows in just three lines from A3, with an instance of T3.1.

T3.8. $\vdash_{ADs} (\sim\mathcal{B} \rightarrow \sim\mathcal{A}) \rightarrow (\mathcal{A} \rightarrow \mathcal{B})$

1. $(\sim\mathcal{B} \rightarrow \sim\mathcal{A}) \rightarrow [(\sim\mathcal{B} \rightarrow \mathcal{A}) \rightarrow \mathcal{B}]$ A3
2. $[(\sim\mathcal{B} \rightarrow \mathcal{A}) \rightarrow \mathcal{B}] \rightarrow [(\mathcal{A} \rightarrow (\sim\mathcal{B} \rightarrow \mathcal{A})) \rightarrow (\mathcal{A} \rightarrow \mathcal{B})]$ T3.4
3. $\mathcal{A} \rightarrow (\sim\mathcal{B} \rightarrow \mathcal{A})$ A1
4. $[(\sim\mathcal{B} \rightarrow \mathcal{A}) \rightarrow \mathcal{B}] \rightarrow (\mathcal{A} \rightarrow \mathcal{B})$ 2,3 T3.6
5. $(\sim\mathcal{B} \rightarrow \sim\mathcal{A}) \rightarrow (\mathcal{A} \rightarrow \mathcal{B})$ 1,4 T3.2

The idea behind this derivation is that the antecedent of A3 is the antecedent of our goal. So we can get the goal by T3.2 with the instance of A3 on (1) and (4). That is, given $(\sim\mathcal{B} \to \sim\mathcal{A}) \to \mathcal{X}$, what we need to get the goal by an application of T3.2 is $\mathcal{X} \to (\mathcal{A} \to \mathcal{B})$. But that is just what (4) is. The challenge is to get (4). Our strategy uses T3.4, and then T3.6 with A1 to "delete" the middle term. This derivation is not particularly easy to see. Here is another approach, which is not all that easy either.

(F)	1.	$(\sim\mathcal{B} \to \sim\mathcal{A}) \to [(\sim\mathcal{B} \to \mathcal{A}) \to \mathcal{B}]$	A3
	2.	$(\sim\mathcal{B} \to \mathcal{A}) \to [(\sim\mathcal{B} \to \sim\mathcal{A}) \to \mathcal{B}]$	1 T3.3
	3.	$\mathcal{A} \to (\sim\mathcal{B} \to \mathcal{A})$	A1
	4.	$\mathcal{A} \to [(\sim\mathcal{B} \to \sim\mathcal{A}) \to \mathcal{B}]$	3,2 T3.2
	5.	$(\sim\mathcal{B} \to \sim\mathcal{A}) \to (\mathcal{A} \to \mathcal{B})$	4 T3.3

This derivation also begins with A3. The idea this time is to use T3.3 to "swing" $\sim\mathcal{B} \to \mathcal{A}$ out, "replace" it by $\mathcal{A}$ with T3.2 and A1, and then use T3.3 to "swing" $\mathcal{A}$ back in.

T3.9. $\vdash_{ADs} \sim\mathcal{A} \to (\mathcal{A} \to \mathcal{B})$

Hint: You can do this in three lines with T3.8 and an instance of A1.

T3.10. $\vdash_{ADs} \sim\sim\mathcal{A} \to \mathcal{A}$

Hint: You can do this in three lines wih instances of T3.7 and T3.9.

T3.11. $\vdash_{ADs} \mathcal{A} \to \sim\sim\mathcal{A}$

Hint: You can do this in three lines with instances of T3.8 and T3.10.

*T3.12. $\vdash_{ADs} (\mathcal{A} \to \mathcal{B}) \to (\sim\sim\mathcal{A} \to \sim\sim\mathcal{B})$

Hint: Use T3.5 and T3.10 to get $(\mathcal{A} \to \mathcal{B}) \to (\sim\sim\mathcal{A} \to \mathcal{B})$; then use T3.4, and T3.11 to get $(\sim\sim\mathcal{A} \to \mathcal{B}) \to (\sim\sim\mathcal{A} \to \sim\sim\mathcal{B})$; the result follows easily by T3.2.

T3.13. $\vdash_{ADs} (\mathcal{A} \to \mathcal{B}) \to (\sim\mathcal{B} \to \sim\mathcal{A})$

Hint: You can do this in three lines with instances of T3.8 and T3.12.

T3.14. $\vdash_{ADs} (\sim\mathcal{A} \rightarrow \mathcal{B}) \rightarrow (\sim\mathcal{B} \rightarrow \mathcal{A})$

Hint: Use T3.4 and T3.10 to get $(\sim\mathcal{B} \rightarrow \sim\sim\mathcal{A}) \rightarrow (\sim\mathcal{B} \rightarrow \mathcal{A})$; the result follows easily with an instance of T3.13.

T3.15. $\vdash_{ADs} (\mathcal{A} \rightarrow \mathcal{B}) \rightarrow [(\sim\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{B}]$

Hint: Use T3.13 and A3 to get $(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow [(\sim\mathcal{B} \rightarrow \mathcal{A}) \rightarrow \mathcal{B}]$; then use T3.5 and T3.14 to get $[(\sim\mathcal{B} \rightarrow \mathcal{A}) \rightarrow \mathcal{B}] \rightarrow [(\sim\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{B}]$; the result follows easily by T3.2.

*T3.16. $\vdash_{ADs} \mathcal{A} \rightarrow [\sim\mathcal{B} \rightarrow \sim(\mathcal{A} \rightarrow \mathcal{B})]$

Hint: Use instances of T3.1 and T3.3 to get $\mathcal{A} \rightarrow [(\mathcal{A} \rightarrow \mathcal{B}) \rightarrow \mathcal{B}]$; then use T3.13 to "turn around" the consequent. This idea of deriving conditionals in "reversed" form, and then using T3.13 or T3.14 to turn them around, is frequently useful for getting tilde outside of a complex expression.

T3.17. $\vdash_{ADs} \mathcal{A} \rightarrow (\mathcal{A} \vee \mathcal{B})$

1.	$\sim\mathcal{A} \rightarrow (\mathcal{A} \rightarrow \mathcal{B})$	T3.9
2.	$\mathcal{A} \rightarrow (\sim\mathcal{A} \rightarrow \mathcal{B})$	1 T3.3
3.	$\mathcal{A} \rightarrow (\mathcal{A} \vee \mathcal{B})$	2 abv

We set as our goal the unabbreviated form. We have this at (2). Then, in the last line, simply observe that the goal abbreviates what has already been shown.

T3.18. $\vdash_{ADs} \mathcal{A} \rightarrow (\mathcal{B} \vee \mathcal{A})$

Hint: Go for $\mathcal{A} \rightarrow (\sim\mathcal{B} \rightarrow \mathcal{A})$. Then, as above, you can get the desired result in one step by abv.

T3.19. $\vdash_{ADs} (\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{B}$

T3.20. $\vdash_{ADs} (\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{A}$

*T3.21. $\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C}) \vdash_{ADs} (\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}$

T3.22. $(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C} \vdash_{ADs} \mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})$

T3.23. $\mathcal{A}, \mathcal{A} \leftrightarrow \mathcal{B} \vdash_{ADs} \mathcal{B}$

Hint: $\mathcal{A} \leftrightarrow \mathcal{B}$ abbreviates the same thing as $(\mathcal{A} \rightarrow \mathcal{B}) \wedge (\mathcal{B} \rightarrow \mathcal{A})$; you may thus move to this expression from $\mathcal{A} \leftrightarrow \mathcal{B}$ by abv.

T3.24. $\mathcal{B}, \mathcal{A} \leftrightarrow \mathcal{B} \vdash_{ADs} \mathcal{A}$

T3.25. $\sim\mathcal{A}, \mathcal{A} \leftrightarrow \mathcal{B} \vdash_{ADs} \sim\mathcal{B}$

T3.26. $\sim\mathcal{B}, \mathcal{A} \leftrightarrow \mathcal{B} \vdash_{ADs} \sim\mathcal{A}$

*E3.2. Provide derivations for T3.6, T3.7, T3.9, T3.10, T3.11, T3.12, T3.13, T3.14, T3.15, T3.16, T3.18, T3.19, T3.20, T3.21, T3.22, T3.23, T3.24, T3.25, and T3.26. As you are working these problems, you may find it helpful to refer to the *AD* summary on p. **??**.

E3.3. For each of the following, expand derivations to include all the steps from theorems. The result should be a derivation in which each step is either a premise, an axiom, or follows from previous lines by a rule. Hint: it may be helpful to proceed in stages as for (C), (D) and then (E) above.

a. Expand your derivation for T3.7.

*b. Expand the above derivation for T3.4.

E3.4. Consider an axiomatic system *A2* which takes $\wedge$ and $\sim$ as primitive operators, and treats $\mathcal{P} \rightarrow \mathcal{Q}$ as an abbreviation for $\sim(\mathcal{P} \wedge \sim\mathcal{Q})$. The axiom schemes are,

A2 A1. $\mathcal{P} \rightarrow (\mathcal{P} \wedge \mathcal{P})$

A2. $(\mathcal{P} \wedge \mathcal{Q}) \rightarrow \mathcal{P}$

A3. $(\mathcal{O} \rightarrow \mathcal{P}) \rightarrow [\sim(\mathcal{P} \wedge \mathcal{Q}) \rightarrow \sim(\mathcal{Q} \wedge \mathcal{O})]$

MP is the only rule. Provide derivations for each of the following, where derivations may appeal to any *prior* result (no matter what *you* have done).

*a. $\mathcal{A} \to \mathcal{B}, \mathcal{B} \to \mathcal{C} \vdash_{A2} \sim(\sim\mathcal{C} \wedge \mathcal{A})$

b. $\vdash_{A2} \sim(\sim\mathcal{A} \wedge \mathcal{A})$

c. $\vdash_{A2} \sim\sim\mathcal{A} \to \mathcal{A}$

*d. $\vdash_{A2} \sim(\mathcal{A} \wedge \mathcal{B}) \to (\mathcal{B} \to \sim\mathcal{A})$

e. $\vdash_{A2} \mathcal{A} \to \sim\sim\mathcal{A}$

f. $\vdash_{A2} (\mathcal{A} \to \mathcal{B}) \to (\sim\mathcal{B} \to \sim\mathcal{A})$

*g. $\sim\mathcal{A} \to \sim\mathcal{B} \vdash_{A2} \mathcal{B} \to \mathcal{A}$

h. $\mathcal{A} \to \mathcal{B} \vdash_{A2} (\mathcal{C} \wedge \mathcal{A}) \to (\mathcal{B} \wedge \mathcal{C})$

*i. $\mathcal{A} \to \mathcal{B}, \mathcal{B} \to \mathcal{C}, \mathcal{C} \to \mathcal{D} \vdash_{A2} \mathcal{A} \to \mathcal{D}$

j. $\vdash_{A2} \mathcal{A} \to \mathcal{A}$

k. $\vdash_{A2} (\mathcal{A} \wedge \mathcal{B}) \to (\mathcal{B} \wedge \mathcal{A})$

l. $\mathcal{A} \to \mathcal{B}, \mathcal{B} \to \mathcal{C} \vdash_{A2} \mathcal{A} \to \mathcal{C}$

m. $\sim\mathcal{B} \to \mathcal{B} \vdash_{A2} \mathcal{B}$

n. $\mathcal{B} \to \sim\mathcal{B} \vdash_{A2} \sim\mathcal{B}$

o. $\vdash_{A2} (\mathcal{A} \wedge \mathcal{B}) \to \mathcal{B}$

p. $\mathcal{A} \to \mathcal{B}, \mathcal{C} \to \mathcal{D} \vdash_{A2} (\mathcal{A} \wedge \mathcal{C}) \to (\mathcal{B} \wedge \mathcal{D})$

q. $\mathcal{B} \to \mathcal{C} \vdash_{A2} (\mathcal{A} \wedge \mathcal{B}) \to (\mathcal{A} \wedge \mathcal{C})$

r. $\mathcal{A} \to \mathcal{B}, \mathcal{A} \to \mathcal{C} \vdash_{A2} \mathcal{A} \to (\mathcal{B} \wedge \mathcal{C})$

s. $\vdash_{A2} [(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C}] \to [\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})]$

t. $\vdash_{A2} [\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})] \to [(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C}]$

*u. $\vdash_{A2} [\mathcal{A} \to (\mathcal{B} \to \mathcal{C})] \to [(\mathcal{A} \wedge \mathcal{B}) \to \mathcal{C})]$

v. $\vdash_{A2} [(\mathcal{A} \wedge \mathcal{B}) \to \mathcal{C}] \to [\mathcal{A} \to (\mathcal{B} \to \mathcal{C})]$

*w. $\mathcal{A} \to \mathcal{B}, \mathcal{A} \to (\mathcal{B} \to \mathcal{C}) \vdash_{A2} \mathcal{A} \to \mathcal{C}$

x. $\vdash_{A2} \mathcal{A} \to [\mathcal{B} \to (\mathcal{A} \wedge \mathcal{B})]$

y. $\vdash_{A2} \mathcal{A} \to (\mathcal{B} \to \mathcal{A})$

Hints: *(i)*: Apply (a) to the first two premises and (f) to the third; then recognize that you have the makings for an application of A3. *(j)*: Apply A1, two instances of (h), and an instance of (i) to get $\mathcal{A} \to ((\mathcal{A} \wedge \mathcal{A}) \wedge (\mathcal{A} \wedge \mathcal{A}))$; the result follows easily with A2 and (i). *(m)*: $\sim\mathcal{B} \to \mathcal{B}$ is equivalent to $\sim(\sim\mathcal{B} \wedge \sim\mathcal{B})$; and $\sim\mathcal{B} \to (\sim\mathcal{B} \wedge \sim\mathcal{B})$ is immediate from A2; you can turn this around by (f) to get $\sim(\sim\mathcal{B} \wedge \sim\mathcal{B}) \to \sim\sim\mathcal{B}$; then it is easy. *(u)*: Use abv so that you are going for $\sim[\mathcal{A} \wedge \sim\sim(\mathcal{B} \wedge \sim\mathcal{C})] \to \sim[(\mathcal{A} \wedge \mathcal{B}) \wedge \sim\mathcal{C}]$; plan on getting to this by (f); the proof then reduces to working from $((\mathcal{A} \wedge \mathcal{B}) \wedge \sim\mathcal{C})$. *(v)*: Structure your proof very much as with (u). *(w)*: Use (u) to set up a "chain" to which you can apply transitivity.

3.3 Quantificational

Chapter 4

Semantics

Having introduced the grammar for our formal languages and even (if you did not skip the last chapter) done derivations in them, we need to say something about *semantics* — about the conditions under which their expressions are true and false. In addition to *logical validity* from chapter 1 and *validity in AD* from chapter 3, this will lead to a third, *semantic* notion of validity. Again, the discussion divides into the relatively simple sentential case, and then the full quantificational version. Recall that we are introducing formal languages in their "pure" form, apart from associations with ordinary language. Having discussed, in this chapter, conditions under which formal expressions are true and not, in the next chapter, we will finally turn to translation, and so to ways formal expressions are associated with ordinary ones.

4.1 Sentential

For any sentential or quantificational language, starting with a sentence and working up its tree, let us say that its *basic* sentences are the first sentences without a truth functional main operator. For a sentential language basic sentences are the sentence letters, as the atomics are precisely the first sentences without a truth functional operator. In the quantificational case, basic sentences may be more complex.[1] In this section, we treat basic sentences as atomic. Our initial focus is on forms with just operators $\sim$ and $\rightarrow$. We begin with an account of the conditions under which sentences are true and not true, learn to apply that account in arbitrary conditions, and

[1]Thus the basic sentences of $A \wedge B$ are just the atomic subformulas A and B. But $Fa \wedge \exists xGx$, has atomic subformulas Fa and Gx, but basic sentences Fa and $\exists xGx$ since the latter does not have a truth functional main operator.

turn to validity. The section concludes with applications to our abbreviations, $\wedge$, $\vee$, and $\leftrightarrow$.

4.1.1 Interpretations and Truth

Sentences are true and false relative to an *interpretation* of basic sentences. In the sentential case, the notion of an interpretation is particularly simple. For any formal language $\mathscr{L}$, a *sentential interpretation* assigns a truth value *true* or *false*, T or F, to each of its basic sentences. Thus, for $\mathscr{L}_s$ we might have interpretations I and J,

(A)

	A	B	C	D	E	F	G	H	
I	T	T	T	T	T	T	T	T	...

	A	B	C	D	E	F	G	H	
J	T	T	F	F	T	T	F	F	...

When a sentence $\mathscr{A}$ is T on an interpretation I, we write I[$\mathscr{A}$] = T, and when it is F, we write, I[$\mathscr{A}$] = F. Thus, in the above case, J[B] = T and J[C] = F.

Truth for complex sentences depends on truth and falsity for their parts. In particular, for any interpretation I,

ST ($\sim$) For any sentence $\mathscr{P}$, I[$\sim\mathscr{P}$] = T iff I[$\mathscr{P}$] = F; otherwise I[$\sim\mathscr{P}$] = F.

($\rightarrow$) For any sentences $\mathscr{P}$ and $\mathscr{Q}$, I[$(\mathscr{P} \rightarrow \mathscr{Q})$] = T iff I[$\mathscr{P}$] = F or I[$\mathscr{Q}$] = T (or both); otherwise I[$(\mathscr{P} \rightarrow \mathscr{Q})$] = F.

Thus a basic sentence is true or false depending on the interpretation. For complex sentences, $\sim\mathscr{P}$ is true iff $\mathscr{P}$ is not true; and $(\mathscr{P} \rightarrow \mathscr{Q})$ is true iff $\mathscr{P}$ is not true or $\mathscr{Q}$ is. It is traditional to represent the information from ST($\sim$) and ST($\rightarrow$) in the following *truth tables*.

T($\sim$)

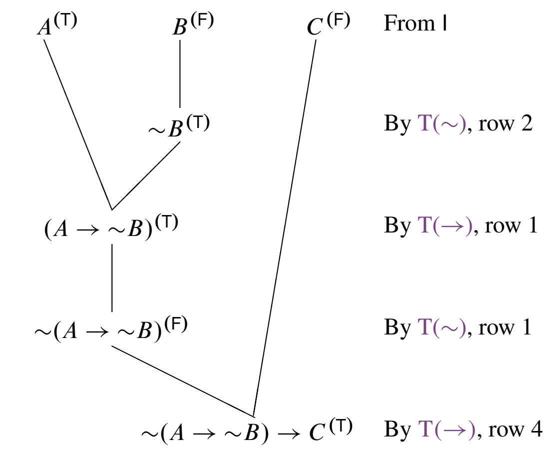
$\mathscr{P}$	$\sim\mathscr{P}$
T	***F***
F	***T***

T($\rightarrow$)

$\mathscr{P}$	$\mathscr{Q}$	$\mathscr{P} \rightarrow \mathscr{Q}$
T	T	***T***
T	F	***F***
F	T	***T***
F	F	***T***

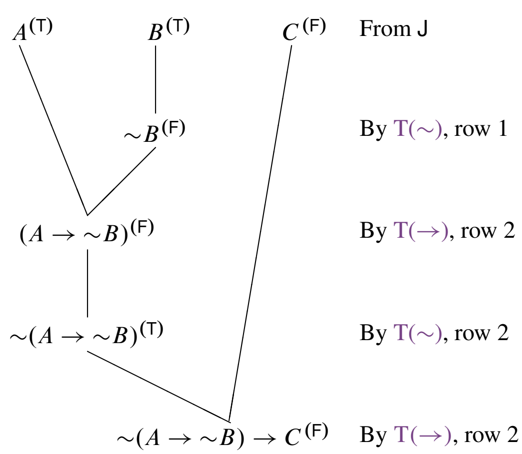
From ST($\sim$), we have that if $\mathscr{P}$ is F then $\sim\mathscr{P}$ is T; and if $\mathscr{P}$ is T then $\sim\mathscr{P}$ is F. This is just the way to read table T($\sim$) from left-to-right in the bottom row, and then the top row. Similarly, from ST($\rightarrow$), we have that $\mathscr{P} \rightarrow \mathscr{Q}$ is T in conditions represented by the first, third and fourth rows of T($\rightarrow$). The only way for $\mathscr{P} \rightarrow \mathscr{Q}$ to be F is when $\mathscr{P}$ is T and $\mathscr{Q}$ is F as in the second row.

ST works recursively. Whether a basic sentence is true comes directly from the interpretation; truth for other sentences depends on truth for their immediate subformulas — and can be read directly off the tables. As usual, we can use trees to see how it works. As we build a the formula from its parts to the whole, so now we calculate truth from the parts to the whole. Suppose $\mathsf{I}[A] = \mathsf{T}$, $\mathsf{I}[B] = \mathsf{F}$, and $\mathsf{I}[C] = \mathsf{F}$. Then $\mathsf{I}[\sim(A \to \sim B) \to C] = \mathsf{T}$.

(B)


The basic tree is the same as the one that shows $\sim(A \to \sim B) \to C$ is a formula. From the interpretation, A is T, B is F, and C is F. These are across the top. Since B is F, from the bottom row of table T(∼), $\sim B$ is T. Since A is T and $\sim B$ is T, reading across the top row of the table T(→), $A \to \sim B$ is T. And similarly, according to the tree, for the rest. You should carefully follow each step.

Here is the same formula considered on another interpretation. Where interpretation J is as on p. 64, $\mathsf{J}[\sim(A \to \sim B) \to C] = \mathsf{F}$.

(C)


This time, for both applications of ST(→), the antecedent is T and the consequent is F; thus we are working on the second row of table T(→), and the conditionals evaluate to F. Again, you should follow each step in the tree.

E4.1. Where the interpretation is as J from p. 64, with J[A] = T, J[B] = T and J[C] = F, use trees to decide whether the following sentences of $\mathcal{L}_s$ are T or F.

*a. $\sim A$

b. $\sim\sim C$

c. $A \rightarrow C$

d. $C \rightarrow A$

*e. $\sim(A \rightarrow A)$

*f. $(\sim A \rightarrow A)$

g. $\sim(A \rightarrow \sim C) \rightarrow C$

h. $(\sim A \rightarrow C) \rightarrow C$

*i. $(A \rightarrow \sim B) \rightarrow \sim(B \rightarrow \sim A)$

j. $\sim(B \rightarrow \sim A) \rightarrow (A \rightarrow \sim B)$

4.1.2 Arbitrary Interpretations

Sentences are true and false relative to an interpretation. But whether an argument is *semantically valid* depends on truth and falsity relative to *every* interpretation. As a first step toward determining semantic validity, in this section, we generalize the method of the last section to calculate truth values relative to arbitrary interpretations.

First, any complex sentence has a *finite* number of basic sentences as components. It is thus possible simply to *list* all the possible interpretations of those basic sentences. If an expression has just one basic sentence $\mathscr{A}$, then on any interpretation whatsoever, that basic sentence must be T or F.

(D)

$\mathscr{A}$
T
F

If an expression has basic sentences $\mathscr{A}$ and $\mathscr{B}$, then the possible interpretations of its basic sentences are,

(E)

$\mathscr{A}$	$\mathscr{B}$
T	T
T	F
F	T
F	F

$\mathscr{B}$ can take its possible values, T and F when $\mathscr{A}$ is true, and $\mathscr{B}$ can take its possible values, T and F when $\mathscr{A}$ is false. And similarly, every time we add a basic sentence, we double the number of possible interpretations, so that n basic sentences always have 2^n possible interpretations. Thus the possible interpretations for three and four basic sentences are,

(F)

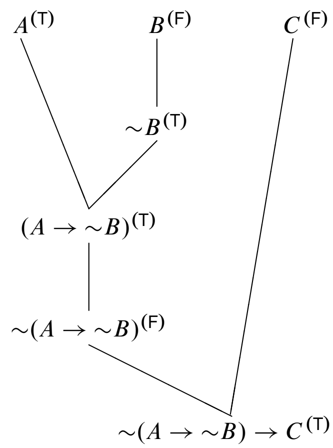
$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

(G)

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{D}$
T	T	T	T
T	T	T	F
T	T	F	T
T	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
F	F	T	T
F	F	T	F
F	F	F	T
F	F	F	F

Extra horizontal lines are added purely for visual convenience. There are $8 = 2^3$ combinations with three basic sentences and $16 = 2^4$ combinations with four. In general, to write down all the possible combinations for n basic sentences, begin by finding the total number $r = 2^n$ of combinations or rows. Then write down a column with half that many $(r/2)$ Ts and half that many $(r/2)$ Fs; then a column alternating half again as many $(r/4)$ Ts and Fs; and a column alternating half again as many $(r/8)$ Ts and Fs — continuing to the n^{th} column alternating groups of just one T and one F. Thus, for example, with four basic sentences, $r = 2^4 = 16$; so we begin with a column consisting of $r/2 = 8$ Ts and $r/2 = 8$ Fs; this is followed by a column alternating groups of 4 Ts and 4 Fs, a column alternating groups of 2 Ts and 2 Fs, and a column alternating groups of 1 T and 1 F. And similarly in other cases.

Given an expression involving, say, four basic sentences, we could imagine doing trees for each of the 16 possible interpretations. But, to exhibit truth values for each of the possible interpretations, we can reduce the amount of work a bit — or at least represent it in a relatively compact form. Suppose $\mathsf{I}[A] = \mathsf{T}$, $\mathsf{I}[B] = \mathsf{F}$, and $\mathsf{I}[C] = \mathsf{F}$, and consider a tree as in (B) from above, along with a "compressed" version of the same information.

(H) 

A	B	C	$\sim$	$(A$	$\to$	$\sim$	$B)$	$\to$	C
T	F	F	F	T	T	T	F	***T***	F

In the table on the right, we begin by simply listing the interpretation we will consider in the lefthand part: A is T, B is F and C is F. Then, under each basic sentence, we put its truth value, and for each formula, we list its truth value *under its main operator*. Notice that the calculation must proceed *precisely* as it does in the tree. It is because B is F, that we put T under the second $\sim$. It is because A is T and $\sim B$ is T that we put a T under the first $\to$. It is because $(A \to \sim B)$ is T that we put F under the first $\sim$. And it is because $\sim(A \to \sim B)$ is F and C is F that we put a T under the second $\to$. In effect, then, we work "down" through the tree, only in this compressed form. We might think of truth values from the tree as "squished" up into the one row. Because there is a T under its main operator, we conclude that the whole formula, $\sim(A \to \sim B) \to C$ is T when I$[A]$ = T, I$[B]$ = F, and I$[C]$ = F. In this way, we might conveniently calculate and represent the truth value of $\sim(A \to \sim B) \to C$ for all eight of the possible interpretations of its basic sentences.

(I)

A	B	C	$\sim$	$(A$	$\to$	$\sim$	$B)$	$\to$	C
T	T	T	T	T	F	F	T	***T***	T
T	T	F	T	T	F	F	T	***F***	F
T	F	T	F	T	T	T	F	***T***	T
T	F	F	F	T	T	T	F	***T***	F
F	T	T	F	F	T	F	T	***T***	T
F	T	F	F	F	T	F	T	***T***	F
F	F	T	F	F	T	T	F	***T***	T
F	F	F	F	F	T	T	F	***T***	F

The emphasized column under the second $\to$ indicates the truth value of $\sim(A \to \sim B) \to C$ for each of the interpretations on the left — which is to say, for every possible interpretation of the three basic sentences. So the only way for $\sim(A \to \sim B) \to C$ to be F is for C to be F, and A and B to be T. Our above tree (H) represents just the fourth row of this table.

In practice, it is easiest to work these *truth tables* "vertically." For this, begin with the basic sentences in some standard order along with all their possible interpretations in the left-hand column. For $\mathcal{L}_s$ let the standard order be alphanumeric ($A, A_1, A_2 \ldots, B, B_1, B_2 \ldots, C \ldots$). And repeat truth values for basic sentences under their occurrences in the formula (this is not crucial, since truth values for basic sentences are already listed on the left; it will be up to you whether to repeat values for basic sentences). This is done in table (J) below.

(J)

A	B	C	$\sim$	$(A$	$\rightarrow$	$\sim$	$B)$	$\rightarrow$	C
T	T	T		T			T		T
T	T	F		T			T		F
T	F	T		T			F		T
T	F	F		T			F		F
F	T	T		F			T		T
F	T	F		F			T		F
F	F	T		F			F		T
F	F	F		F			F		F

(K)

A	B	C	$\sim$	$(A$	$\rightarrow$	$\sim$	$B)$	$\rightarrow$	C
T	T	T		T		F	T		T
T	T	F		T		F	T		F
T	F	T		T		T	F		T
T	F	F		T		T	F		F
F	T	T		F		F	T		T
F	T	F		F		F	T		F
F	F	T		F		T	F		T
F	F	F		F		T	F		F

Now, given the values for B as in (J), we are in a position to calculate the values for $\sim B$; so get the T($\sim$) table in you mind, put your eye on the column under B in the formula (or on the left if you have decided not to repeat the values for B under its occurrence in the formula). Then fill in the column under the second $\sim$, reversing the values from under B. This is accomplished in (K). Given the values for A and $\sim B$, we are now in a position to calculate values for $A \rightarrow \sim B$; so get the T($\rightarrow$) table in your head, and put your eye on the columns under A and $\sim B$. Then fill in the column

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It is worth asking what happens if basic sentences are listed in some order other than alphanumeric.

A	B
T	T
T	F
F	T
F	F

B	A
T	T
T	F
F	T
F	F

All the combinations are still listed, but their locations in a table change.

Each of the above tables lists all of the combinations for the basic sentences. But the first table has the interpretation I with I[A] = T and I[B] = F in the second row, where the second table has this combination in the third. Similarly, the tables exchange rows for the interpretation J with J[A] = F and J[B] = T. As it turns out, the only real consequence of switching rows is that it becomes difficult to compare tables as, for example, with the back of the book. And it may matter as part of the standard of correctness for exercises!

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under the first →, going with F only when A is T and $\sim B$ is F. This is accomplished in (L).

(L)

A	B	C	$\sim$	$(A$	$\rightarrow$	$\sim$	$B)$	$\rightarrow$	C
T	T	T		T	F	F	T		T
T	T	F		T	F	F	T		F
T	F	T		T	T	T	F		T
T	F	F		T	T	T	F		F
F	T	T		F	T	F	T		T
F	T	F		F	T	F	T		F
F	F	T		F	T	T	F		T
F	F	F		F	T	T	F		F

(M)

A	B	C	$\sim$	$(A$	$\rightarrow$	$\sim$	$B)$	$\rightarrow$	C
T	T	T	T	T	F	F	T		T
T	T	F	T	T	F	F	T		F
T	F	T	F	T	T	T	F		T
T	F	F	F	T	T	T	F		F
F	T	T	F	F	T	F	T		T
F	T	F	F	F	T	F	T		F
F	F	T	F	F	T	T	F		T
F	F	F	F	F	T	T	F		F

Now we are ready to fill in the column under the first $\sim$. So get the T($\sim$) table in your head, and put your eye on the column under the first →. The column is completed in table (M). And the table is finished as in (I) by completing the column under the last →, based on the columns under the first $\sim$ and under the C. Notice again, that the order in which you work the columns exactly parallels the order from the tree.

As another example, consider these tables for $\sim(B \rightarrow A)$, the first with truth values repeated under basic sentences, the second without.

(N)

A	B	$\sim$	$(B$	$\rightarrow$	$A)$
T	T	***F***	T	T	T
T	F	***F***	F	T	T
F	T	***T***	T	F	F
F	F	***F***	F	T	F

(O)

A	B	$\sim$	$(B$	$\rightarrow$	$A)$
T	T	***F***		T	
T	F	***F***		T	
F	T	***T***		F	
F	F	***F***		T	

We complete the table as before. First, with our eye on the columns under B and A, we fill in the column under →. Then, with our eye on that column, we complete the one under $\sim$. For this, first, notice that $\sim$ is the *main* operator. You would *not* calculate $\sim B$ and then the arrow! Rather, your calculations move from the smaller parts to the larger; so the arrow comes first and then the tilde. Again, the order is the same as on a tree. Second, if you do not repeat values for basic formulas, be careful about $B \rightarrow A$; the leftmost column of table (O), under A, is the column for the *consequent* and the column immediately to its right, under B, is for the *antecedent*; in this case, then, the second row under arrow is T and the *third* is F. Though it is fine to omit columns under basic sentences, as they are already filled in on the left side, you should *not* skip other columns, as they are essential building blocks for the final result.

E4.2. For each of the following sentences of $\mathcal{L}_s$ construct a truth table to determine its truth value for each of the possible interpretations of its basic sentences.

*a. $\sim\sim A$

b. $\sim(A \to A)$

c. $(\sim A \to A)$

*d. $(\sim B \to A) \to B$

e. $\sim(B \to \sim A) \to B$

f. $(A \to \sim B) \to \sim(B \to \sim A)$

*g. $C \to (A \to B)$

h. $[A \to (C \to B)] \to [(A \to C) \to (A \to B)]$

*i. $(\sim A \to B) \to (\sim C \to D)$

j. $\sim(A \to \sim B) \to \sim(C \to \sim D)$

4.1.3 Validity

As we have seen, sentences are true and false relative to an interpretation. For any interpretation, a complex sentence has some definite value. But whether an argument is *sententially valid* depends on truth and falsity relative to *every* interpretation. Suppose a formal argument has premises $\mathcal{P}_1 \dots \mathcal{P}_n$ and conclusion $\mathcal{Q}$. Then,

> $\mathcal{P}_1 \dots \mathcal{P}_n$ *sententially entail* $\mathcal{Q}$ ($\mathcal{P}_1 \dots \mathcal{P}_n \vDash_s \mathcal{Q}$) iff there is no sentential interpretation I such that $\mathsf{I}[\mathcal{P}_1] = \mathsf{T}$ and ... and $\mathsf{I}[\mathcal{P}_n] = \mathsf{T}$ but $\mathsf{I}[\mathcal{Q}] = \mathsf{F}$.

We can put this more generally as follows. Suppose Γ (Gamma) is a set of formulas — these are the premises. Say $\mathsf{I}[\Gamma] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{P}] = \mathsf{T}$ for each $\mathcal{P}$ in Γ. Then,

> SV Γ *sententially entails* $\mathcal{Q}$ ($\Gamma \vDash_s \mathcal{Q}$) iff there is no sentential interpretation I such that $\mathsf{I}[\Gamma] = \mathsf{T}$ but $\mathsf{I}[\mathcal{Q}] = \mathsf{F}$.

Where the members of Γ are $\mathcal{P}_1 \dots \mathcal{P}_n$, this says the same thing as before. Γ sententially entails $\mathcal{Q}$ when there is no sentential interpretation that makes each member of Γ true and $\mathcal{Q}$ false. If Γ sententially entails $\mathcal{Q}$ we say the *argument* whose premises are the members of Γ and conclusion is $\mathcal{Q}$ is *sententially valid*. Γ does not sententially entail $\mathcal{Q}$ ($\Gamma \nvDash_s \mathcal{Q}$) when there is some sentential interpretation on which all the members of Γ are true, but $\mathcal{Q}$ is false. We can think of the premises as *constraining* the interpretations that matter: for validity it is just the interpretations where the members of Γ are all true, on which the conclusion $\mathcal{Q}$ cannot be false. If

Γ has no members then there are no constraints on relevant interpretations, and the conclusion must be true on every interpretation in order for it to be valid. In this case where there are no premises, we simply write $\vDash_s \mathcal{Q}$, and if $\mathcal{Q}$ is valid it is *logically true* (a *tautology*). Notice the new double turnstile $\vDash$ for this semantic notion, in contrast to the single turnstile $\vdash$ for derivations from chapter 3.

Given that we are already in a position to exhibit truth values for arbitrary interpretations, it is a simple matter to determine whether an argument is sententially valid. Where the premises and conclusion of an argument include basic sentences $\mathcal{B}_1 \ldots \mathcal{B}_n$, begin by calculating the truth values of the premises and conclusion for each of the possible interpretations for $\mathcal{B}_1 \ldots \mathcal{B}_n$. Then *look* to see if any interpretation makes all the premises true but the conclusion false. If no interpretation makes the premises true and the conclusion not, then by SV, the argument is sententially valid. If some interpretation does make the premises true and the conclusion false, then it is not valid.

Thus, for example, suppose we want to know whether the following argument is sententially valid.

$$\text{(P)}\quad \begin{array}{l} (\sim A \to B) \to C \\ B \\ \hline C \end{array}$$

By SV, the question is whether there is an interpretation that makes the premises true and the conclusion not. So we begin by calculating the values of the premises and conclusion for each of the possible interpretations of the basic sentences in the premises and conclusion.

A	B	C	($\sim$	A	$\to$	B)	$\to$	C	B	/ C
T	T	T	F	T	T	T	***T***	T	***T***	***T***
T	T	F	F	T	T	T	***F***	F	***T***	***F***
T	F	T	F	T	T	F	***T***	T	***F***	***T***
T	F	F	F	T	T	F	***F***	F	***F***	***F***
F	T	T	T	F	T	T	***T***	T	***T***	***T***
F	T	F	T	F	T	T	***F***	F	***T***	***F***
F	F	T	T	F	F	F	***T***	T	***F***	***T***
F	F	F	T	F	F	F	***T***	F	***F***	***F***

Now we simply look to see whether any interpretation makes all the premises true but the conclusion not. Interpretations represented by the top row, ones that make A, B, and C all T, do not make the premises true and the conclusion not, because both the premises and the conclusion come out true. In the second row, the conclusion is false, but the first premise is false as well; so not *all* the premises are true *and* the conclusion is false. In the third row, we do not have either all the premises true or the conclusion false. In the fourth row, though the conclusion is false, the premises are not true. In the fifth row, the premises are true, but the conclusion is not false. In the sixth row, the first premise is not true, and in the seventh and eighth rows, the second premise is not true. So no interpretation makes the premises true and the conclusion false. So by SV, $(\sim A \to B) \to C,\ B \vDash_s C$. Notice that the only column that matters for a complex formula is the one under its main operator — the one that gives the value of *the sentence* for each of the interpretations; the other columns exist only to support the calculation of the value of the whole.

In contrast, $\sim[(B \to A) \to B] \nvDash_s \sim(A \to B)$. That is, an argument with premise, $\sim[(B \to A) \to B]$ and conclusion $\sim(A \to B)$ is not sententially valid.

(Q)

A	B	$\sim$	$[(B$	$\to$	$A)$	$\to$	$B]$	/	$\sim$	$(A$	$\to$	$B)$	
T	T	***F***	T	T	T	T	T		***F***	T	T	T	
T	F	***T***	F	T	T	F	F		***T***	T	F	F	
F	T	***F***	T	F	F	T	T		***F***	F	T	T	
F	F	***T***	F	T	F	F	F		***F***	F	T	F	$\Leftarrow$

In the first row, the premise is F. In the second, the conclusion is T. In the third, the premise is F. However, in the last, the premise is T, and the conclusion is F. So there are interpretations (any interpretation that makes A and B both F) that make the premise T and the conclusion not true. So by SV, $\sim[(B \to A) \to B] \nvDash_s \sim(A \to B)$, and the argument is not sententially valid. All it takes is *one* interpretation that makes all the premises T and the conclusion F to render an argument not sententially valid. Of course, there might be more than one, but one is enough!

As a final example, consider table (I) for $\sim(A \to \sim B) \to C$ on p. 68 above. From the table, there is an interpretation where the sentence is not true. Thus, by SV, $\nvDash_s \sim(A \to \sim B) \to C$. A sentence is valid only when it is true on every interpretation. Since there is an interpretation on which it is not true, the sentence is not valid (not a logical truth).

Since all it takes to demonstrate invalidity is *one* interpretation on which all the premises are true and the conclusion is false, we do not actually need an entire table to demonstrate invalidity. You may decide to produce a whole truth table in order to *find* an interpretation to demonstrate invalidity. But we can sometimes work "backward" from what we are trying to show to an interpretation that does the job. Thus, for example, to find the result from table (Q), we need an interpretation on which the premise is T and the conclusion is F. That is, we need a row like this,

(R)

A	B	$\sim$	$[(B$	$\to$	$A)$	$\to$	$B]$	/	$\sim$	$(A$	$\to$	$B)$
		T							***F***			

In order for the premise to be T, the conditional in the brackets must be F. And in order for the conclusion to be F, the conditional must be T. So we can fill in this much.

(S)

A	B	$\sim$	$[(B$	$\to$	$A)$	$\to$	$B]$	/	$\sim$	$(A$	$\to$	$B)$
		T				F			***F***		T	

Since there are three ways for an arrow to be T, there is not much to be done with the conclusion. But since the conditional in the premise is F, we know that its antecedent is T and consequent is F. So we have,

(T)

A	B	$\sim$	$[(B$	$\to$	$A)$	$\to$	$B]$	/	$\sim$	$(A$	$\to$	$B)$
		T		T		F	F		***F***		T	

That is, if the conditional in the brackets is F, then $(B \to A)$ is T and B is F. But now we can fill in the information about B wherever it occurs. The result is as follows.

(U)

A	B	$\sim$	$[(B$	$\to$	$A)$	$\to$	$B]$	/	$\sim$	$(A$	$\to$	$B)$
	F	***T***	F	T		F	F		***F***		T	F

Since the first B in the premise is F, the first conditional in the premise is T irrespective of the assignment to A. But, with B false, the only way for the conditional in the argument's conclusion to be T is for A to be false as well. The result is our completed row.

(V)

A	B	$\sim$	$[(B$	$\to$	$A)$	$\to$	$B]$	/	$\sim$	$(A$	$\to$	$B)$
F	F	***T***	F	T	F	F	F		***F***	F	T	F

And we have recovered the row that demonstrates invalidity — without doing the entire table. In this case, the full table had only four rows, and we might just as well have done the whole thing. However, when there are many rows, this "shortcut" approach can be attractive. A disadvantage is that sometimes it is not obvious just *how* to proceed. In this example each stage led to the next. At stage (S), there were three ways to make the conclusion true. We were able to proceed insofar as the premise forced the next step. But it might have been that neither the premise nor the conclusion forced a definite next stage. In this sort of case, you might decide to do the whole table, just so that you can grapple with all the different combinations in an orderly way.

Notice what happens when we try this approach with an argument that is not invalid. Returning to argument (P) above, suppose we try to find a row where the premises are T and the conclusion is F. That is, we set out to find a row like this,

(W)

A	B	C	$(\sim$	A	$\to$	$B)$	$\to$	C	B	/	C
							T		***T***		***F***

Immediately, we are in a position to fill in values for B and C.

(X)

A	B	C	$(\sim$	A	$\to$	$B)$	$\to$	C	B	/	C
	T	F				T	***T***	F	***T***		***F***

Since the first premise is a true arrow with a false consequent, its antecedent $(\sim A \to B)$ must be F. But this requires that $\sim A$ be T and that B be F.

(Y)

A	B	C	$(\sim$	A	$\rightarrow$	$B)$	$\rightarrow$	C	B	/	C
	T	F	T		F	F/T	***T***	F	***T***		***F***

And there is no way to set B to F, as we have already seen that it has to be T in order to keep the second premise true — and no interpretation makes B *both* T and F. At this stage, we know, in our hearts, that there is no way to make both of the premises true and the conclusion false. In Part II we will turn this knowledge into an official mode of reasoning for validity. However, for now, let us consider a single row of a truth table (or a marked row of a full table) sufficient to demonstrate *invalidity*, but require a full table, exhibiting all the options, to show that an argument is sententially valid.

You may encounter odd situations where premises are never T, where conclusions are never F, or whatever. But if you stick to the definition, always asking whether there is any interpretation of the basic sentences that makes all the premises T and the conclusion F, all will be well.

E4.3. For each of the following, use truth tables to decide whether the entailment claims hold. Notice that a couple of the tables are already done from E4.2.

*a. $A \rightarrow \sim A \vDash_s \sim A$

b. $\sim A \rightarrow A \vDash_s \sim A$

*c. $A \rightarrow B, \sim A \vDash_s \sim B$

d. $A \rightarrow B, \sim B \vDash_s \sim A$

e. $\sim(A \rightarrow \sim B) \vDash_s B$

f. $\vDash_s C \rightarrow (A \rightarrow B)$

*g. $\vDash_s [A \rightarrow (C \rightarrow B)] \rightarrow [(A \rightarrow C) \rightarrow (A \rightarrow B)]$

h. $(A \rightarrow B) \rightarrow \sim(B \rightarrow A), \sim A, \sim B \vDash_s \sim(C \rightarrow C)$

i. $[A \rightarrow \sim(B \rightarrow \sim C)], [B \rightarrow (\sim C \rightarrow D)] \vDash_s A \rightarrow \sim(B \rightarrow \sim D)$

j. $\sim[(A \rightarrow \sim(B \rightarrow \sim C)) \rightarrow D], \sim D \rightarrow A \vDash_s C$

4.1.4 Abbreviations

We turn finally to applications for our abbreviations. Consider, first, a truth table for $\mathcal{P} \vee \mathcal{Q}$, that is for $\sim\mathcal{P} \rightarrow \mathcal{Q}$.

T(∨)

$\mathcal{P}$	$\mathcal{Q}$	$\sim$	$\mathcal{P}$	$\rightarrow$	$\mathcal{Q}$
T	T	F	T	***T***	T
T	F	F	T	***T***	F
F	T	T	F	***T***	T
F	F	T	F	***F***	F

so that

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \vee \mathcal{Q}$
T	T	***T***
T	F	***T***
F	T	***T***
F	F	***F***

When $\mathcal{P}$ is T and $\mathcal{Q}$ is T, $\mathcal{P} \vee \mathcal{Q}$ is T; when $\mathcal{P}$ is T and $\mathcal{Q}$ is F, $\mathcal{P} \vee \mathcal{Q}$ is T; and so forth. Thus, when $\mathcal{P}$ is T and $\mathcal{Q}$ is T, we *know* that $\mathcal{P} \vee \mathcal{Q}$ is T, without going through all the steps to get there in the unabbreviated form. Just as when $\mathcal{P}$ is a formula and $\mathcal{Q}$ is a formula, we move directly to the conclusion that $\mathcal{P} \vee \mathcal{Q}$ is a formula without explicitly working all the intervening steps, so if we know the truth value of $\mathcal{P}$ and the truth value of $\mathcal{Q}$, we can move in a tree by the above table to the truth value of $\mathcal{P} \vee \mathcal{Q}$ without all the intervening steps. And similarly for the other abbreviating sentential operators. For $\wedge$,

T(∧)

$\mathcal{P}$	$\mathcal{Q}$	$\sim$	$(\mathcal{P}$	$\rightarrow$	$\sim$	$\mathcal{Q})$
T	T	***T***	T	F	F	T
T	F	***F***	T	T	T	F
F	T	***F***	F	T	F	T
F	F	***F***	F	T	T	F

so that

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \wedge \mathcal{Q}$
T	T	***T***
T	F	***F***
F	T	***F***
F	F	***F***

And for $(\leftrightarrow)$,

T(↔)

$\mathcal{P}$	$\mathcal{Q}$	$\sim$	$[(\mathcal{P}$	$\rightarrow$	$\mathcal{Q})$	$\rightarrow$	$\sim$	$(\mathcal{Q}$	$\rightarrow$	$\mathcal{P})]$
T	T	***T***	T	T	T	F	F	T	T	T
T	F	***F***	T	F	F	T	F	F	T	T
F	T	***F***	F	T	T	T	T	T	F	F
F	F	***T***	F	T	F	F	F	F	T	F

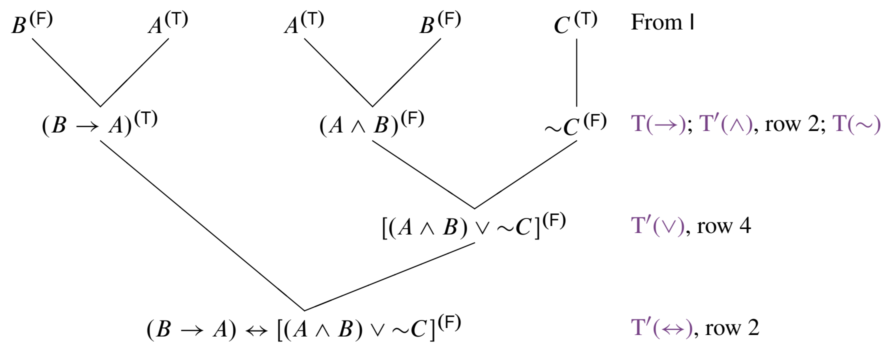
so that

$\mathcal{P}$	$\mathcal{Q}$	$\mathcal{P} \leftrightarrow \mathcal{Q}$
T	T	***T***
T	F	***F***
F	T	***F***
F	F	***T***

As a help toward remembering these tables, notice that $\mathcal{P} \vee \mathcal{Q}$ is F only when $\mathcal{P}$ is F and $\mathcal{Q}$ is F; $\mathcal{P} \wedge \mathcal{Q}$ is T only when $\mathcal{P}$ is T and $\mathcal{Q}$ is T; and $\mathcal{P} \leftrightarrow \mathcal{Q}$ is T only when $\mathcal{P}$ and $\mathcal{Q}$ are the *same* and F when $\mathcal{P}$ and $\mathcal{Q}$ are different. We can think of these clauses as representing derived clauses T′(∨), T′(∧), and T′(↔) to the definition for truth.

And nothing prevents direct application of the derived tables in trees. Suppose, for example, $\mathsf{I}[A] = \mathsf{T}$, $\mathsf{I}[B] = \mathsf{F}$, and $\mathsf{I}[C] = \mathsf{T}$. Then $\mathsf{I}[(B \rightarrow A) \leftrightarrow [(A \wedge B) \vee \sim C] = \mathsf{F}$.

(Z)



We might get the same result by working through the full tree for the unabbreviated form. But there is no need. When A is T and B is F, we *know* that $(A \wedge B)$ is F; when $(A \wedge B)$ is F and $\sim C$ is F, we *know* that $[(A \wedge B) \vee C]$ is F; and so forth. Thus we move through the tree directly by the derived tables.

Similarly, we can work directly with abbreviated forms in truth tables.

(AA)

A	B	C	$(B$	$\rightarrow$	$A)$	$\leftrightarrow$	$[(A$	$\wedge$	$B)$	$\vee$	$\sim$	$C]$
T	T	T	T	T	T	***T***	T	T	T	T	F	T
T	T	F	T	T	T	***T***	T	T	T	T	T	F
T	F	T	F	T	T	***F***	T	F	F	F	F	T
T	F	F	F	T	T	***T***	T	F	F	T	T	F
F	T	T	T	F	F	***T***	F	F	T	F	F	T
F	T	F	T	F	F	***F***	F	F	T	T	T	F
F	F	T	F	T	F	***F***	F	F	F	F	F	T
F	F	F	F	T	F	***T***	F	F	F	T	T	F

Tree (Z) represents just the third row of this table. As before, we construct the table "vertically," with tables for abbreviating operators in mind as appropriate.

Finally, given that we have tables for abbreviated forms, we can use them for evaluation of *arguments* with abbreviated forms. Thus, for example, $A \leftrightarrow B$, $A \vDash_s$

There are a couple of different ways tables for our operators can be understood: First, as we shall see in **??**, it is possible to take tables for operators other than $\sim$ and $\rightarrow$ as basic, say, just T($\sim$) and T($\vee$), or just T($\sim$) and T($\wedge$), and then abbreviate $\rightarrow$ in terms of them. Challenge: What expression involving just $\sim$ and $\vee$ has the same table as $\rightarrow$? what expression involving just $\sim$ and $\wedge$? Another option is to introduce all five as basic. Then the task is not *showing* that the table for $\vee$ is TTTF — that is given; rather we simply notice that $\mathcal{P} \vee \mathcal{Q}$, say, is redundant with $\sim\mathcal{P} \rightarrow \mathcal{Q}$. Again, our approach with $\sim$ and $\rightarrow$ basic has the advantage of preserving relative simplicity in the basic language (though other minimal approaches would do so as well).

$A \wedge B$.

(AB)

A	B	$(A \leftrightarrow B)$	A	/	$(A \wedge B)$
T	T	T ***T*** T	***T***		T ***T*** T
T	F	T ***F*** F	***T***		T ***F*** F
F	T	F ***F*** T	***F***		F ***F*** T
F	F	F ***T*** F	***F***		F ***F*** F

There is no row where each of the premises is true and the conclusion is false. So the argument is sententially valid. And, from either of the following rows,

(AC)

A	B	C	D	$[(B \rightarrow A) \wedge (\sim C \vee D)]$	$[(A \leftrightarrow \sim D) \wedge (\sim D \rightarrow B)]$	/	B
F	F	T	T	F T F ***T*** F T T T	F T FT ***T*** F T T F		***F***
F	F	F	T	F T F ***T*** T F T T	F T FT ***T*** F T T F		***F***

we may conclude that $[(B \rightarrow A) \wedge (\sim C \vee D)]$, $[(A \leftrightarrow \sim D) \wedge (\sim D \rightarrow B)] \not\models_s B$. In this case, the shortcut table is attractive relative to the full version with sixteen rows!

E4.4. For each of the following, use truth tables to decide whether the entailment claims hold.

a. $\models_s A \vee \sim A$

b. $A \leftrightarrow [\sim A \leftrightarrow (A \wedge \sim A)]$, $A \rightarrow \sim(A \leftrightarrow A) \models_s \sim A \rightarrow A$

*c. $B \vee \sim C \models_s B \rightarrow C$

*d. $A \vee B$, $\sim C \rightarrow \sim A$, $\sim(B \wedge \sim C) \models_s C$

e. $A \rightarrow (B \vee C)$, $C \leftrightarrow B$, $\sim C \models_s \sim A$

f. $\sim(A \wedge \sim B) \models_s \sim A \vee B$

g. $A \wedge (B \rightarrow C) \models_s (A \wedge B) \vee (A \wedge C)$

*h. $\models_s \sim(A \leftrightarrow B) \leftrightarrow (A \wedge \sim B)$

i. $A \vee (B \wedge \sim C)$, $\sim(\sim B \vee C) \rightarrow \sim A \models_s \sim A \leftrightarrow \sim(C \vee \sim B)$

j. $A \vee B$, $\sim D \rightarrow (C \vee A) \models_s B \leftrightarrow \sim C$

E4.5. For each of the following, use truth tables to decide whether the entailment claims hold. Hint: the trick here is to identify the *basic* sentences. After that, everything proceeds in the usual way with truth values assigned to the basic sentences.

Semantics Quick Reference (Sentential)

For any formal language $\mathscr{L}$, a *sentential interpretation* assigns a truth value *true* or *false*, T or F, to each of its basic sentences. Then for any interpretation I,

ST ($\sim$) For any sentence $\mathscr{P}$, I[$\sim\mathscr{P}$] = T iff I[$\mathscr{P}$] = F; otherwise I[$\sim\mathscr{P}$] = F.

($\rightarrow$) For any sentences $\mathscr{P}$ and $\mathscr{Q}$, I[($\mathscr{P} \rightarrow \mathscr{Q}$)] = T iff I[$\mathscr{P}$] = F or I[$\mathscr{Q}$] = T (or both); otherwise I[($\mathscr{P} \rightarrow \mathscr{Q}$)] = F.

And for abbreviated expressions,

ST′ ($\wedge$) For any sentences $\mathscr{P}$ and $\mathscr{Q}$, I[($\mathscr{P} \wedge \mathscr{Q}$)] = T iff I[$\mathscr{P}$] = T and I[$\mathscr{Q}$] = T; otherwise I[($\mathscr{P} \wedge \mathscr{Q}$)] = F.

($\vee$) For any sentences $\mathscr{P}$ and $\mathscr{Q}$, I[($\mathscr{P} \vee \mathscr{Q}$)] = T iff I[$\mathscr{P}$] = T or I[$\mathscr{Q}$] = T (or both); otherwise I[($\mathscr{P} \vee \mathscr{Q}$)] = F.

($\leftrightarrow$) For any sentences $\mathscr{P}$ and $\mathscr{Q}$, I[($\mathscr{P} \leftrightarrow \mathscr{Q}$)] = T iff I[$\mathscr{P}$] = I[$\mathscr{Q}$]; otherwise I[($\mathscr{P} \leftrightarrow \mathscr{Q}$)] = F.

If Γ (Gamma) is a set of formulas, I[Γ] = T iff I[$\mathscr{P}$] = T for each $\mathscr{P}$ in Γ. Then, where the members of Γ are the formal premises of an argument, and sentence $\mathscr{P}$ is its conclusion,

SV Γ *sententially entails* $\mathscr{P}$ iff there is no sentential interpretation I such that I[Γ] = T but I[$\mathscr{P}$] = F.

We treat a single row of a truth table (or a marked row of a full table) as sufficient to demonstrate *invalidity*, but require a full table, exhibiting all the options, to show that an argument is sententially valid.

*a. $\exists xAx \rightarrow \exists xBx, \sim\exists xAx \vDash_s \exists xBx$

b. $\forall xAx \rightarrow \sim\exists x(Ax \wedge \forall yBy), \exists x(Ax \wedge \forall yBy) \vDash_s \sim\forall xAx$

E4.6. For each of the following concepts, explain in an essay of about two pages, so that (high-school age) Hannah could understand. In your essay, you should (i) identify the objects to which the concept applies, (ii) give and explain the definition, and give and explicate examples of your own construction (iii) where the concept applies, and (iv) where it does not. Your essay should exhibit an understanding of methods from the text.

a. Sentential interpretations and truth for complex sentences.

b. Sentential validity.

4.2 Quantificational

Chapter 5

Translation

We have introduced logical validity from chapter 1, along with notions of semantic validity from chapter 4, and validity in an axiomatic derivation system from chapter 3. But logical validity applies to arguments expressed in ordinary language, where the other notions apply to arguments expressed in a formal language. Our guiding idea has been to *use* the formal notions with application to ordinary arguments via *translation* from ordinary language to the formal ones. It is to the translation task that we now turn. After some general discussion, we will take up issues specific to the sentential, and then the quantificational, cases.

5.1 General

As speakers of ordinary languages (at least English for those reading this book) we presumably have some understanding of the conditions under which ordinary language sentences are true and false. Similarly, we now have an understanding of the conditions under which sentences of our formal languages are true and false. This puts us in a position to recognize when the conditions under which ordinary sentences are true are the *same* as the conditions under which formal sentences are true. And that is what we want: Our goal is to translate the premises and conclusion of ordinary arguments into formal expressions that are true when the ordinary sentences are true, and false when the ordinary sentences are false. Insofar as validity has to do with conditions under which sentences are true and false, our translations should thus be an adequate basis for evaluations of validity.

We can put this point with greater precision. Formal sentences are true and false relative to interpretations. As we have seen, many different interpretations of a formal language are possible. In the sentential case, any sentence letter can be true or false

— so that there are 2^n ways to interpret any n sentence letters. When we specify an interpretation, we select just one of the many available options. Thus, for example, we might set $\mathsf{I}[B] = \mathsf{T}$ and $\mathsf{I}[H] = \mathsf{F}$. But we might also specify an interpretation as follows,

(A) B: Bill is happy
H: Hillary is happy

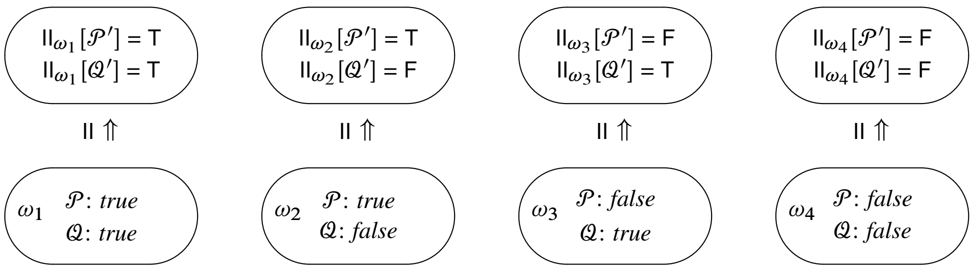
intending B to take the same truth value as 'Bill is happy' and H the same as 'Hillary is happy'. In this case, the single specification might result in different interpretations, depending on how the world is: Depending on how Bill and Hillary are, the interpretation of B might be true or false, and similarly for H. That is, specification (A) is really a *function* from ways the world could be (from complete and consistent stories) to interpretations of the sentence letters. It results in a specific or *intended* interpretation relative to any way the world could be. Thus, where ω (omega) ranges over ways the world could be, (A) is a function II which results in an intended interpretation II_ω corresponding to any such way — thus $\mathsf{II}_\omega[B]$ is T if Bill is happy at ω and F if he is not.

When we set out to translate some ordinary sentences into a formal language, we always begin by specifying an intended interpretation of the formal language for arbitrary ways the world can be. In the sentential case, this typically takes the form of a specification like (A). Then for any way the world can be ω there is an intended interpretation II_ω of the formal language. Given this, for an ordinary sentence $\mathcal{A}$, the aim is to produce a formal counterpart $\mathcal{A}'$ such that $\mathsf{II}_\omega[\mathcal{A}'] = \mathsf{T}$ iff the ordinary $\mathcal{A}$ is true in world ω. This is the content of saying we want to produce formal expressions that "are true when the ordinary sentences are true, and false when the ordinary sentences are false." In fact, we can turn this into a *criterion of goodness* for translation.

CG Given some ordinary sentence $\mathcal{A}$, a translation consisting of an interpretation function II and formal sentence $\mathcal{A}'$ is *good* iff it captures available sentential/quantificational structure and, where ω is any way the world can be, $\mathsf{II}_\omega[\mathcal{A}'] = \mathsf{T}$ iff $\mathcal{A}$ is true at ω.

If there is a collection of sentences, a translation is *good* given an II where *each* member $\mathcal{A}$ of the collection of sentences has an $\mathcal{A}'$ such that $\mathsf{II}_\omega[\mathcal{A}'] = \mathsf{T}$ iff $\mathcal{A}$ is true at ω. Set aside the question of what it is to capture "available" sentential/quantificational structure, this will emerge as we proceed. For now, the point is simply that we want formal sentences to be true on intended interpretations when originals are true at corresponding worlds, and false on intended interpretations when originals are false. CG says that this correspondence is necessary for goodness. And, supposing that sufficient structure is reflected, according to CG such correspondence is sufficient as well.

The situation might be pictured as follows. There is a specification II which results in an intended interpretation corresponding to any way the world can be. And corresponding to ordinary sentences $\mathcal{P}$ and $\mathcal{Q}$ there are formal sentences $\mathcal{P}'$ and $\mathcal{Q}'$. Then,



The interpretation function results in an intended interpretation corresponding to each world. The translation is good only if no matter how the world is, the values of $\mathcal{P}'$ and $\mathcal{Q}'$ on the intended interpretations match the values of $\mathcal{P}$ and $\mathcal{Q}$ at the corresponding worlds or stories.

The premises and conclusion of an argument are some sentences. So the translation of an argument is *good* iff the translation of the sentences that are its premises and conclusion is good. And good translations of arguments put us in a position to *use* our machinery to evaluate questions of validity. Of course, so far, this is an abstract description of what we are about to do. But it should give some orientation, and help you understand what is accomplished as we proceed.

5.2 Sentential

We begin with the sentential case. Again, the general idea is to *recognize* when the conditions under which ordinary sentences are true are the *same* as the conditions under which formal ones are true. Surprisingly perhaps, the hardest part is on the side of recognizing truth conditions in ordinary language. With this in mind, let us begin with some definitions whose application is to expressions of *ordinary* language; after that, we will turn to a procedure for translation, and to discussion of particular operators.

5.2.1 Some Definitions

In this section, we introduce a series of definitions whose application is to ordinary language. These definitions are not meant to compete with anything you have learned in English class. They are rather specific to our purposes. With the definitions under our belt, we will be able to say with some precision what we want to do.

First, a *declarative sentence* is a sentence which has a truth value. ‘Snow is white’ and ‘Snow is green’ are declarative sentences — the first true and the second false. ‘Study harder!’ and ‘Why study?’ are sentences, but not declarative sentences. Given this, a *sentential operator* is an expression containing “blanks” such that when the blanks are filled with declarative sentences, the result is a declarative sentence. In ordinary speech and writing, such blanks do not typically appear (!) however punctuation and expression typically fill the same role. Examples are,

John believes that ____

John heard that ____

It is not the case that ____

____ and ____

‘John believes that snow is white’, ‘John believes that snow is green’, and ‘John believes that dogs fly’ are all sentences — some more plausibly true than others. Still, ‘Snow is white’, ‘Snow is green’, and ‘Dogs fly’ are all declarative sentences, and when we put them in the blank of ‘John believes that ____’ the result is a declarative sentence, where the same would be so for any declarative sentence in the blank; so ‘John believes that ____’ is a sentential operator. Similarly, ‘Snow is white and dogs fly’ is a declarative sentence — a false one, since dogs do not fly. And, so long as we put declarative sentences in the blanks of ‘____ and ____’ the result is always a declarative sentence. So ‘____ and ____’ is a sentential operator. In contrast,

When ____

____ is white ____

are not sentential operators. Though ‘Snow is white’ is a declarative sentence, ‘When snow is white’ is an adverbial clause, not a declarative sentence. And, though ‘Dogs fly’ and ‘Snow is green’ are declarative sentences, ‘Dogs fly is white snow is green’ is ungrammatical nonsense. If you can think of even one case where putting declarative sentences in the blanks of an expression does not result in a declarative sentence, then the expression is not a sentential operator. So these are not sentential operators.

Now, as in these examples, we can think of some declarative sentences as generated by the combination of sentential operators with other declarative sentences. Declarative sentences generated from other sentences by means of sentential operators are *compound*; all others are *simple*. Thus, for example, 'Bob likes Mary' and 'Socrates is wise' are simple sentences, they do not have a declarative sentence in the blank of any operator. In contrast, 'John believes that Bob likes Mary' and 'Jim heard that John believes that Bob likes Mary' are compound. The first has a simple sentence in the blank of 'John believes that ____'. The second puts a compound in the blank of 'Jim heard that ____'.

For cases like these, the *main operator* of a compound sentence is that operator not in the blank of any other operator. The main operator of 'John believes that Bob likes Mary' is 'John believes that ____'. And the main operator of 'Jim heard that John believes that Bob likes Mary' is 'Jim heard that ____'. The main operator of 'It is not the case that Bob likes Sue and it is not the case that Sue likes Bob' is '____ and ____', for that is the operator not in the blank of any other. Notice that the main operator of a sentence need not be the *first* operator in the sentence. Observe also that operator structure may not be obvious. Thus, for example, 'Jim heard that Bob likes Sue and Sue likes Jim' is capable of different interpretations. It might be, 'Jim heard that Bob likes Sue and Sue likes Jim' with main operator, 'Jim heard that ____' and the compound, 'Bob likes Sue and Sue likes Jim' in its blank. But it might be 'Jim heard that Bob likes Sue and Sue likes Jim' with main operator, '____ and ____'. The question is what Jim heard, and what the 'and' joins. As suggested above, punctuation and expression often serve in ordinary language to disambiguate confusing cases. These questions of interpretation are not peculiar to our purposes! Rather they are the ordinary questions that might be asked about what one is saying. The underline structure serves to disambiguate claims, to make it very clear how the operators apply.

When faced with a compound sentence, the best approach is start with the whole, rather than the parts. So begin with blank(s) for the main operator. Thus, as we have seen, the main operator of 'It is not the case that Bob likes Sue, and it is not the case that Sue likes Bob' is '____ and ____'. So begin with lines for that operator, 'It is not the case that Bob likes Sue and it is not the case that Sue likes Bob' (leaving space for lines above). Now focus on the sentence in one of the blanks, say the left; that sentence, 'It is not the case that Bob likes Sue' is is a compound with main operator, 'it is not the case that ____'. So add the underline for that operator, 'It is not the case that Bob likes Sue and it is not the case that Sue likes Bob'. The sentence in the blank of 'it is not the case that ____' is simple. So turn to the sentence in the right blank of the main operator. That sentence has main operator 'it is not the case that ____'. So add an underline. In this way we end up with, 'It is not the case that Bob likes Sue and it is not the case that Sue likes Bob' where, again, the sentence in the last blank is simple. Thus, a complex problem is reduced to ones that are progressively more simple. Perhaps this problem was obvious from the start. But this approach will serve you well as problems get more complex!

We come finally to the key notion of a *truth functional* operator. A sentential operator is *truth functional* iff any compound generated by it has its truth value wholly determined by the truth values of the sentences in its blanks. We will say that the truth value of a compound is "determined" by the truth values of sentences in blanks just in case there is no way to switch the truth value of the whole while keeping truth values of sentences in the blanks constant. This leads to a test for truth functionality: We show that an operator is *not* truth functional, if we come up with some situation(s) where truth values of sentences in the blanks are the same, but the truth value of the resulting compounds are not. To take a simple case, consider 'John believes that ____'. If things are pretty much as in the actual world, 'Dogs fly' and 'There is a Santa' are both false. But if John is a small child it may be that,

Dogs fly
(B) John believes that There is a Santa
F/T ***F***

the compound is false with one in the blank, and true with the other. Thus the truth value of the compound is not wholly determined by the truth value of the sentence in the blank. We have found a situation where sentences with the same truth value in the blank result in a different truth value for the whole. Thus 'John believes that ____' is not truth functional. We might make the same point with a pair of sentences that are true, say 'Dogs bark' and 'There are infinitely many prime numbers' (be clear in your mind about how this works).

As a second example, consider, '____ because ____'. Suppose 'You are happy', 'You got a good grade', 'There are fish in the sea' and 'You woke up this morning' are all true.

You are happy — You got a good grade
(C) There are fish in the sea because You work up this morning
T ***T/F*** ***T***

Still, it is natural to think that, the truth value of the compound, 'You are happy because you got a good grade' is true, but 'There are fish in the sea because you woke up this morning' is false. For perhaps getting a good grade makes you happy, but the fish in the sea have nothing to do with your waking up. Thus there are consistent situations or stories where sentences in the blanks have the same truth values, but the compounds do not. Thus, by the definition, '____ because ____' is not a truth functional operator. To show that an operator is not truth functional it is sufficient to produce some situation of this sort: where truth values for sentences in the blanks match, but truth values for the compounds do not. Observe that sentences in the blanks are *fixed* but the value of the compound is not. Thus, it would be enough to find, say, a case where sentences in the first blank are T, sentences in the second are F but the value of the whole flips from T to F. To show that an operator is not truth functional, any matching combination that makes the whole switch value will do.

To show that an operator is truth functional, we need to show that no such cases are possible. For this, we show *how* the truth value of what is in the blank determines the truth value of the whole. As an example, consider first,

	It is not the case that	____
(D)	***F***	T
	T	F

In this table, we represent the truth value of whatever is in the blank by the column under the blank, and the truth value for the whole by the column under the operator. If we put something true according to a consistent story into the blank, the resultant compound is sure to be false according to that story. Thus, for example, in the true story, 'Snow is white', '$2+2=4$' and 'Dogs bark' are all true; correspondingly, 'It is not the case that snow is white', 'It is not the case that $2+2=4$' and 'It is not the case that dogs bark' are all false. Similarly, if we put something false according to a story into the blank, the resultant compound is sure to be true according to the story. Thus, for example, in the true story, 'Snow is green' and '$2+2=3$' are both false. Correspondingly, 'It is not the case that snow is green' and 'It is not the case that $2+2=3$' are both true. It is no coincidence that the above table for 'It is not the case that ____' looks like the table for $\sim$. We will return to this point shortly.

For a second example of a truth functional operator, consider '____ and ____'. This seems to have table,

	____	and	____
	T	***T***	T
(E)	T	***F***	F
	F	***F***	T
	F	***F***	F

Consider a situation where Bob and Sue each love themselves, but hate each other. Then Bob loves Bob and Sue loves Sue is true. But if at least one blank has a sentence that is false, the compound is false. Thus, for example, in that situation, Bob loves Bob and Sue loves Bob is false; Bob loves Sue and Sue loves Sue is false; and Bob loves Sue and Sue loves Bob is false. For a compound, '____ and ____' to be true, the sentences in both blanks have to be true. And if they are both true, the compound is itself true. So the operator is truth functional. Again, it is no coincidence that the table looks so much like the table for $\wedge$. To show that an operator is truth functional, it is sufficient to produce the table that shows how the truth values of the compound are fixed by the truth values of the sentences in the blanks.

Definitions for Translation

DC A *declarative sentence* is a sentence which has a truth value.

SO A *sentential operator* is an expression containing "blanks" such that when the blanks are filled with declarative sentences, the result is a declarative sentence.

CS Declarative sentences generated from other sentences by means of sentential operators are *compound*; all others are *simple*.

MO The *main operator* of a compound sentence is that operator not in the blank of any other operator.

TF A sentential operator is *truth functional* iff any compound generated by it has its truth value wholly determined by the truth values of the sentences in its blanks.

To show that an operator is not truth functional it is sufficient to produce some situation where truth values for sentences in the blanks are constant, but truth values for the compounds are not.

To show that an operator is truth functional, it is sufficient to produce the table that shows how the truth values of the compound are fixed by truth values of the sentences in the blanks.

For an interesting sort of case, consider the operator 'According to every consistent story ____', and the following attempted table,

	According to every consistent story	____
(F)	?	T
	F	F

(On some accounts, this operator works like 'Necessarily ____'). Say we put some sentence $\mathscr{P}$ that is false according to a consistent story into the blank. Then since $\mathscr{P}$ is false according to that very story, it is not the case that $\mathscr{P}$ according to every consistent story — and the compound is sure to be false. So we fill in the bottom row under the operator as above. So far, so good. But consider 'Dogs bark' and '2+2 = 4'. Both are true according to the true story. But only the second is true according to *every* consistent story. So the compound is false with the first in the blank, true with the second. So 'According to every consistent story ____' is therefore *not* a truth functional operator. The truth value of the compound is not *wholly* determined by the truth value of the sentence in the blank. Similarly, it is natural to think that '____ because ____' is false whenever one of the sentences in its blanks is false. It cannot be true that $\mathscr{P}$ because $\mathscr{Q}$ if not-$\mathscr{P}$, and it cannot be true that $\mathscr{P}$ because $\mathscr{Q}$ if not-$\mathscr{Q}$. If you are not happy, then it cannot be that you are happy because you understand the material; and if you do not understand the material, it cannot be that you are happy because you understand the material. So far, then, the table for '____ because ____' is like the table for '____ and ____'.

(G)

____	because	____
T	?	T
T	***F***	F
F	***F***	T
F	***F***	F

However, as we saw just above, in contrast to '____ and ____', compounds generated by '____ because ____' may or may not be true when sentences in the blanks are both true. So, although '____ and ____' is truth functional, '____ because ____' is not.

Thus the question is whether we can complete a table of the above sort: If there is a way to complete the table, the operator is truth functional. The test to show an operator is not truth functional simply finds some case to show that such a table cannot be completed.

E5.1. For each of the following, identify the simple sentences that are parts. If the sentence is compound, use underlines to exhibit its operator structure, and say what is its main operator.

a. Bob likes Mary.

b. Jim believes that Bob likes Mary.

c. It is not the case that Bob likes Mary.

d. Jane heard that it is not the case that Bob likes Mary.

e. Jane heard that Jim believes that it is not the case that Bob likes Mary.

f. Voldemort is very powerful, but it is not the case that Voldemort kills Harry at birth.

g. Harry likes his godfather and Harry likes Dumbledore, but it is not the case that Harry likes his uncle.

*h. Hermoine believes that studying is good, and Hermione studies hard, but Ron believes studying is good, and it is not the case that Ron studies hard.

i. Malfoy believes mudbloods are scum, but it is not the case that mudbloods are scum; and Malfoy is a dork.

j. Harry believes that Voldemort is evil and Hermione believes that Voldemort is evil, but it is not the case that Bellatrix believes that Voldemort is evil.

E5.2. Which of the following operators are truth functional and which are not? If the operator is truth functional, display the relevant table; if it is not, give cases that flip the value of the compound, with the value in the blanks constant.

*a. It is a fact that ____

b. Elmore believes that ____

*c. ____ but ____

d. According to some consistent story ____

e. Although ____, ____

*f. It is always the case that ____

g. Sometimes it is the case that ____

h. ____ therefore ____

i. ____ however ____

j. Either ____ or ____ (or both)

5.2.2 Parse Trees

We are now ready to outline a procedure for translation into our formal sentential language. In the end, you will often be able to see how translations should go and to write them down without going through all the official steps. However, the procedure should get you thinking in the right direction, and remain useful for complex cases. To translate some ordinary sentences $\mathcal{P}_1 \dots \mathcal{P}_n$ the basic translation procedure is,

TP (1) Convert the ordinary $\mathcal{P}_1 \dots \mathcal{P}_n$ into corresponding ordinary equivalents exposing truth functional and operator structure.

(2) Generate a "parse tree" for each of $\mathcal{P}_1 \dots \mathcal{P}_n$ and specify the interpretation function II by assigning sentence letters to sentences at the bottom nodes.

(3) Using sentence letters from II and equivalent formal operators, construct a parallel tree that translates each node from the parse tree, to generate a formal $\mathcal{P}'_i$ for each $\mathcal{P}_i$.

For now at least, the idea behind step (1) is simple: Sometimes all you need to do is expose operator structure by introducing underlines. In complex cases, this can be difficult! But we know how to do this. Sometimes, however, truth functional structure does not lie on the surface. Ordinary sentences are *equivalent* when they are true and false in exactly the same consistent stories. And we want ordinary equivalents exposing truth functional structure. Suppose $\mathcal{P}$ is a sentence of the sort,

(H) Bob is not happy

Is this a truth functional compound? Not officially. There is no declarative sentence in the blank of a sentential operator; so it is not compound; so it is not a truth functional compound. But one might think that (H) is short for,

(I) It is not the case that Bob is happy

which is a truth functional compound. At least, (H) and (I) are equivalent in the sense that they are true and false in the same consistent stories. Similarly, 'Bob and Carol are happy' is not a compound of the sort we have described, because 'Bob' is not a declarative sentence. However, it is a short step from this sentence to the equivalent, 'Bob is happy and Carol is happy' which is an official truth functional compound. As we shall see, in some cases, this step can be more complex. But let us leave it at that for now.

Moving to step (2), in a *parse tree* we begin with sentences constructed as in step (1). If a sentence has a *truth functional* main operator, then it branches downward for the sentence(s) in its blanks. If these have truth functional main operators, they branch for the sentences in *their* blanks; and so forth, until sentences are simple or have non-truth functional main operators. Then we construct the interpretation function II by assigning a distinct sentence letter to each distinct sentence at a bottom node from a tree for the original $\mathcal{P}_1 \dots \mathcal{P}_n$.

Some simple examples should make this clear. Say we want to translate a collection of four sentences.

1. Bob is happy

2. Carol is not happy

3. Bob is healthy and Carol is not

4. Bob is happy and John believes that Carol is not healthy

The first is a simple sentence. Thus there is nothing to be done at step (1). And since there is no main operator, there is no branching and the sentence itself is a completed parse tree. The tree is just,

(J) Bob is happy

Insofar as the simple sentence is a complete branch of the tree, it counts as a bottom node of its tree. It is not yet assigned a sentence letter, so we assign it one. B_1: Bob is happy. We select this letter to remind us of the assignment.

The second sentence is not a truth functional compound. Thus in the first stage, 'Carol is not happy' is expanded to the equivalent, 'It is not the case that Carol is happy'. In this case, there is a main operator; since it is truth functional, the tree has some structure.

(K)

It is not the case that Carol is happy

Carol is happy

The bottom node is simple, so the tree ends. 'Carol is happy' is not assigned a letter; so we assign it one. C_1: Carol is happy.

The third sentence is equivalent to, Bob is healthy and it is not the case that Carol is healthy. Again, the operators are truth functional, and the result is a structured tree.

(L)

Bob is healthy and it is not the case that Carol is healthy

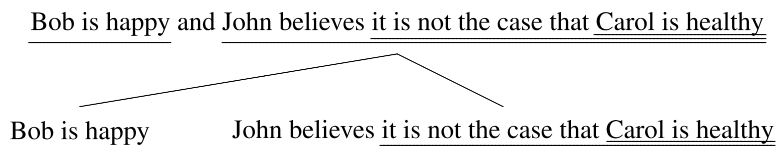
Bob is healthy — it is not the case that Carol is healthy

Carol is healthy

The main operator is truth functional. So there is a branch for each of the sentences in its blanks. Observe that underlines continue to reflect the structure of *these* sentences (so we "lift" the sentences from their blanks with structure intact). On the left, 'Bob is healthy' has no main operator, so it does not branch. On the right, 'it is not the case that Carol is healthy' has a truth functional main operator, and so branches. At bottom, we end up with 'Bob is healthy' and 'Carol is healthy'. Neither has a letter, so we assign them ones. B_2: Bob is healthy; C_2: Carol is healthy.

The final sentence is equivalent to, Bob is happy and John believes it is not the case that Carol is healthy. It has a truth functional main operator. So there is a structured tree.

(M)



On the left, 'Bob is happy' is simple. On the right, 'John believes it is not the case that Carol is healthy' is complex. But its main operator is not truth functional. So *it does not branch*. We only branch for sentences in the blanks of truth functional main operators. Given this, we proceed in the usual way. 'Bob is happy' already has a letter. The other does not; so we give it one. J: John believes it is not the case that Carol is healthy.

And that is all. We have now compiled an interpretation function,

II B_1: Bob is happy

C_1: Carol is happy

B_2: Bob is healthy

C_2: Carol is healthy

J: John believes it is not the case that Carol is healthy

Of course, we might have chosen different letters. All that matters is that we have a distinct letter for each distinct sentence. Our intended interpretations are ones that capture available sentential structure, and make the sentence letters true in situations where these sentences are true and false when they are not. In the last case, there is a compulsion to think that we can somehow get down to the simple sentence 'Carol is happy'. But resist temptation! A non-truth functional operator "seals off" that upon which it operates, and forces us to treat the compound as a unit. We do not automatically assign sentence letters to simple sentences, but rather to parts that are *not* truth functional compounds. Simple sentences fit this description. But so do compounds with non-truth-functional main operators.

E5.3. Use our method to expose truth functional structure and produce parse trees for each of the following. Use your trees to produce an interpretation function for the sentences. Hint: pay attention to punctuation as a guide to structure.

a. Bingo is spotted, and Spot can play bingo.

b. Bingo is not spotted, and Spot cannot play bingo.

c. Bingo is spotted, and believes that Spot cannot play bingo.

*d. It is not the case that: Bingo is spotted and Spot can play bingo.

e. It is not the case that: Bingo is not spotted and Spot cannot play bingo.

E5.4. Use our method to expose truth functional structure and produce parse trees for each of the following. Use your trees to produce an interpretation function for the sentences.

*a. People have rights and dogs have rights, but rocks do not.

b. It is not the case that: rocks have rights, but people do not.

c. Aliens believe that rocks have rights, but it is not the case that people believe it.

d. Aliens landed in Roswell NM in 1947, and live underground but not in my backyard.

e. Rocks do not have rights and aliens do not have rights, but people and dogs do.

5.2.3 Formal Sentences

Now we are ready for step (3) of the translation procedure TP. Our aim is to generate translations by constructing a parallel tree where the force of ordinary truth functional operators is captured by *equivalent* formal operators. An ordinary truth functional operator has a table. Similarly, our formal expressions have tables. Say an ordinary truth functional operator is *equivalent* to some formal expression containing blanks just in case their tables are the same. Thus '∼____' is equivalent to 'it is not the case that ____'. They are equivalent insofar as in each case, the whole has the opposite truth value of what is in the blank. Similarly, '____ ∧ ____' is equivalent to '____ and ____'. In either case, when sentences in the blanks are both T the whole is T, and in other cases, the whole is F. Of course, the complex '∼(____ → ∼____)' takes the same values as the '____ ∧ ____' that abbreviates it. So different formal expressions may be equivalent to a given ordinary one.

To see how this works, let us return to the sample sentences from above. Again, the idea is to generate a parallel tree. We begin by *using* the sentence letters from our interpretation function for the bottom nodes. The case is particularly simple when the tree has no structure. 'Bob is happy' had a simple unstructured tree, and we assigned it a sentence letter directly. Thus our original and parallel trees are,

(N) Bob is happy B_1

So for a simple sentence, we simply read off the final translation from the interpretation function. So much for the first sentence.

As we have seen, the second sentence is equivalent to 'It is not the case that Carol is happy' with a parse tree as on the left below. We begin the parallel tree on the other side.

(O)

It is not the case that Carol is happy
|
Carol is happy C_1

We know how to translate the bottom node. But now we want to capture the force of the truth functional operator with some equivalent formal operator(s). For this, we need a formal expression containing blanks whose table mirrors the table for the sentential operator in question. In this case, '∼____' works fine. That is, we have,

∼	____		It is not the case that	____
F	T		***F***	T
T	F		***T***	F

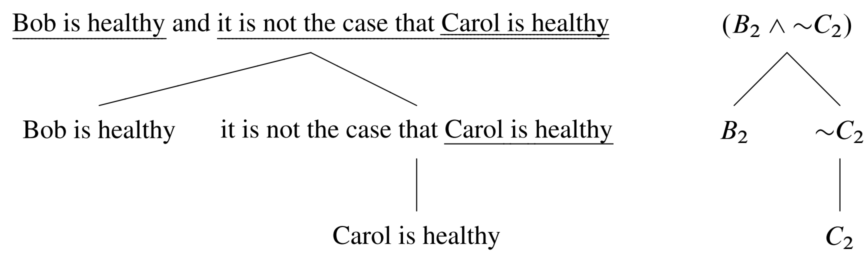
In each case, when the expression in the blank is T, the whole is F, and when the expression in the blank is F, the whole is T. So '∼____' is sufficient as a translation of 'It is not the case that ____'. Other formal expressions might do just as well. Thus, for example, we might go with, '∼∼∼____'. The table for this is the same as the table for '∼____'. But it is hard to see why we would do this, with ∼ so close at hand. Now the idea is to apply the equivalent operator *to* the already translated expression from the blank. But this is easy to do. Thus we complete the parallel tree as follows.

It is not the case that Carol is happy $\sim C_1$
| |
Carol is happy C_1

The result is the completed translation, $\sim C_1$.

The third sentence has a parse tree as on the left, and resultant parallel tree as on the right. As usual, we begin with sentence letters from the interpretation function for the bottom nodes.

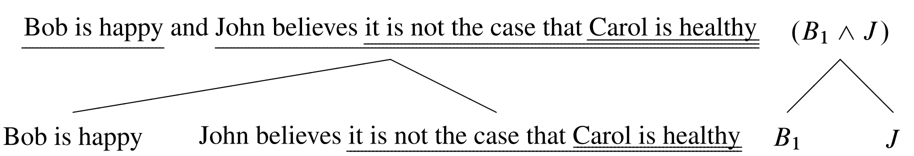
(P)



Given translations for the bottom nodes, we work our way through the tree, applying equivalent operators to translations already obtained. As we have seen, a natural translation of 'it is not the case that ____' is '$\sim$____'. Thus, working up from 'Carol is healthy', our parallel to 'it is not the case that Carol is healthy' is $\sim C_2$. But now we have translations for both of the blanks of '____ and ____'. As we have seen, this has the same table as '(____ $\wedge$ ____)'. So that is our translation. Again, other expressions might do. In particular, $\wedge$ is an abbreviation with the same table as '$\sim$(____ $\rightarrow$ $\sim$____)'. In each case, the whole is true when the sentences in both blanks are true, and otherwise false. Since this is the same as for '____ and ____', either would do as a translation. But again, the simplest thing is to go with '(____ $\wedge$ ____)'. Thus the final result is $(B_2 \wedge \sim C_2)$. With the alternate translation for the main operator, the result would have been $\sim(B_2 \rightarrow \sim\sim C_2)$. Observe that the parallel tree is an upside-down version of the (by now quite familiar) tree by which we would show that the expression is a sentence.

Our last sentence is equivalent to, Bob is happy and John believes it is not the case that Carol is healthy. Given what we have done, the parallel tree should be easy to construct.

(Q)



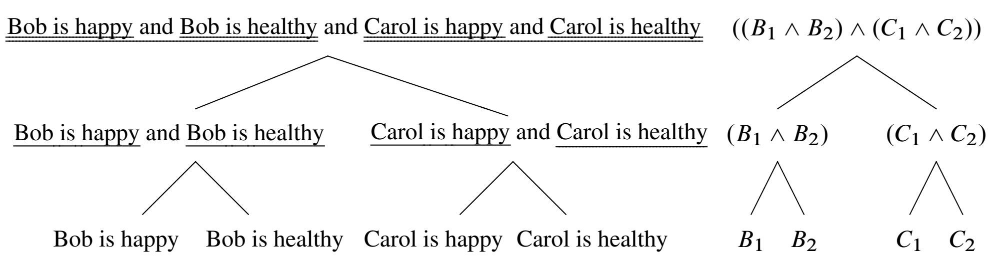
Given that the tree "bottoms out" on both 'Bob is happy' and 'John believes it is not the case that Carol is healthy' the only operator to translate is the main operator '____ and ____'. And we have just seen how to deal with that. The result is the completed translation, $(B_1 \wedge J)$.

Again, once you become familiar with this procedure, the full method, with the trees, may become tedious — and we will often want to set it to the side. But notice: the method breeds good habits! And the method puts us in a position to translate complex expressions, even ones that are so complex that we can barely grasp what they are saying. Beginning with the main operator, we break expressions down from complex parts to ones that are simpler. Then we construct translations, one operator at a time, where each step is manageable. Also, we should be able to see *why* the method results in good translations: For any situation and corresponding intended interpretation, truth values for *basic* parts are the same by the specification of the interpretation function. And given that operators are equivalent, truth values for parts built out of them must be the same as well, all the way up to the truth value of the whole. We satisfy the first part of our criterion CG insofar as the way we break down sentences in parse trees forces us to capture all the truth functional structure there is to be captured.

For a last example, consider, 'Bob is happy and Bob is healthy and Carol is happy and Carol is healthy'. This is true only if 'Bob is happy', 'Bob is healthy', 'Carol is happy', and 'Carol is healthy' are all true. But the method may apply in different ways. We might at step one, treat the sentence as a complex expression involving multiple uses of '____ and ____'; perhaps something like,

(R) Bob is happy and Bob is healthy and Carol is happy and Carol is healthy

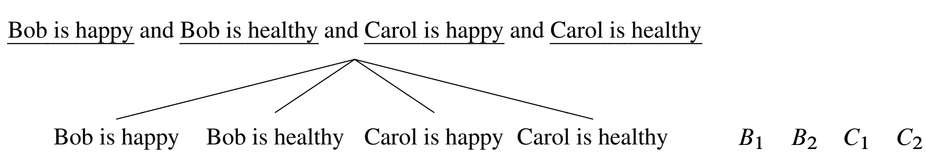
In this case, there is a straightforward move from the ordinary operators to formal ones in the final step. That is, the situation is as follows.



So we use multiple applications of our standard caret operator. But we might have treated the sentence as something like,

(S) Bob is happy and Bob is healthy and Carol is happy and Carol is healthy

involving a single four-blank operator, '____ and ____ and ____ and ____', which yields true only when sentences in all its blanks are true. We have not seen anything like this before, but nothing stops a tree with four branches all at once. In this case, we would begin,



But now, for an equivalent operator we need a formal expression with *four* blanks that is true when sentences in all the blanks are true and otherwise false. Here is something that would do: '((____ ∧ ____) ∧ (____ ∧ ____))'. On either of these approaches, then, the result is $((B_1 \wedge B_2) \wedge (C_1 \wedge C_2))$. Other options might result in something like $(((B_1 \wedge B_2) \wedge C_1) \wedge C_2)$. In this way, there is room for shifting burden between steps one and three. Such shifting explains how step (1) can be more complex than it was initially represented to be. Choices about expanding truth functional structure in the initial stage may matter for what are the equivalent operators at the end. And the case exhibits how there are options for different, equally good, translations of the same ordinary expressions. What matters for CG is that resultant expressions capture available structure and be true when the originals are true and false when the originals are false. In most cases, one translation will be more *natural* than others, and it is good form to strive for natural translations. If there had been a comma so that the original sentence was, 'Bob is happy and Bob is healthy, and Carol is happy and Carol is healthy' it would have been most natural to go for an account along the lines of (R). And it is crazy to use, say, '∼∼∼____' when '∼____' will do as well.

*E5.5. Construct parallel trees to complete the translation of the sentences from E5.3 and E5.4. Hint: you will not need any operators other than ∼ and ∧.

E5.6. Use our method to translate each of the following. That is, generate parse trees with an interpretation function for all the sentences, and then parallel trees to produce formal equivalents.

a. Plato and Aristotle were great philosophers, but Ayn Rand was not.

b. Plato was a great philosopher, and everything Plato said was true, but Ayn Rand was not a great philosopher, and not everything she said was true.

*c. It is not the case that: everything Plato, and Aristotle, and Ayn Rand said was true.

d. Plato was a great philosopher but not everything he said was true, and Aristotle was a great philosopher but not everything he said was true.

e. Not everyone agrees that Ayn Rand was not a great philosopher, and not everyone thinks that not everything she said was true.

E5.7. Use our method to translate each of the following. That is, generate parse trees with an interpretation function for all the sentences, and then parallel trees to produce formal equivalents.

a. Bob and Sue and Jim will pass the class.

b. Sue will pass the class, but it is not the case that: Bob will pass and Jim will pass.

c. It is not the case that: Bob will pass the class and Sue will not.

d. Jim will not pass the class, but it is not the case that: Bob will not pass and Sue will not pass.

e. It is not the case that: Jim will pass and not pass, and it is not the case that: Sue will pass and not pass.

5.2.4 And, Or, Not

Our idea has been to recognize when truth conditions for ordinary and formal sentences are the same. As we have seen, this turns out to require recognizing when *operators* have the same tables. We have had a lot to say about 'it is not the case that ____' and '____ and ____'. We now turn to a more general treatment. We will not be able to provide a complete menu of ordinary operators. Rather, we will see that some uses of some ordinary operators can be appropriately translated by our symbols. We should be able to discuss enough cases for you to see how to approach others on a case-by-case basis. The discussion is organized around our operators, $\sim$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$, taken in that order.

First, as we have seen, 'It is not the case that ____' has the same table as $\sim$. And various ordinary expressions may be equivalent to expressions involving this operator. Thus, 'Bob is not married' and 'Bob is unmarried' might be understood as equivalent to 'It is not the case that <u>Bob is married</u>'. Given this, we might assign a sentence letter, say, M to 'Bob is married' and translate $\sim M$. But the second case calls for comment. By comparison, consider, 'Bob is unlucky'. Given what we have done, it is natural to treat 'Bob is unlucky' as equivalent to 'It is not the case that <u>Bob is lucky</u>'; assign L to 'Bob is lucky'; and translate $\sim L$. But this is not obviously right. Consider three situations: (i) Bob goes to Las Vegas with $1,000, and comes away with $1,000,000. (ii) Bob goes to Las Vegas with $1,000, and comes away with $100, having seen a show and had a good time. (iii) Bob goes to Las Vegas with $1,000, falls into a manhole on his way into the casino, and has his money stolen by a light-fingered thief on the way down. In the first case he is lucky; in the third, unlucky. But, in the second, one might want to say that he was neither lucky nor unlucky.

(i) Bob is lucky
(ii) Bob is neither lucky nor unlucky
(iii) Bob is unlucky
} It is not the case that Bob is lucky

If this is right, 'Bob is unlucky' is *not* equivalent to 'It is not the case that Bob is lucky' — for it is not the case that Bob is lucky in *both* situations (ii) and (iii). Thus we might have to assign 'Bob is lucky' one letter, and 'Bob is unlucky' another.[1] Decisions about this sort of thing may depend heavily on context, and assumptions which are in the background of conversation. We will ordinarily *assume* contexts where there is no "neutral" state — so that being unlucky is not being lucky, and similarly in other cases.

Second, as we have seen, '____ and ____' has the same table as $\wedge$. As you may recall from E5.2, another common operator that works this way is '____ but ____'. Consider, for example, 'Bob likes Mary but Mary likes Jim'. Suppose Bob does like Mary and Mary likes Jim; then the compound sentence is true. Suppose one of the simples is false, Bob does not like Mary or Mary does not like Jim; then the compound is false. Thus '____ but ____' has the table,

(T)

____	but	____
T	***T***	T
T	***F***	F
F	***F***	T
F	***F***	F

and so has the same table as $\wedge$. So, in this case, we might assign B to 'Bob likes Mary' M to 'Mary likes Jim', and translate, $(B \wedge M)$. Of course, the ordinary expression 'but' carries a sense of opposition that 'and' does not. Our point is not that 'and' and 'but' somehow *mean* the same, but rather that compounds formed by means of them are true and false under the same truth functional conditions. Another common operator with this table is 'Although ____, ____'. You should convince yourself that this is so, and be able to find other ordinary terms that work just the same way.

Once again, however, there is room for caution in some cases. Consider, for example, 'Bob took a shower and got dressed'. Given what we have done, it is natural to treat this as equivalent to 'Bob took a shower and Bob got dressed'; assign letters S and D; and translate $(S \wedge D)$. But this is not obviously right. Suppose Bob gets

[1] Or so we have to do in the context of our logic where T and F are the only truth values. Another option is to allow three values so that the one letter might be T, F or neither. It is possible to proceed on this basis — though the two valued (classical) approach has the virtue of relative simplicity! With the classical approach as background, some such alternatives are developed in Priest, *Non-Classical Logics*.

dressed, but then realizes that he is late for a date and forgot to shower, so he jumps in the shower fully clothed, and air-dries on the way. Then it is true that Bob took a shower, and true that Bob got dressed. But is it true that Bob took a shower and got dressed? If not — because the order is wrong — our translation $(S \wedge D)$ might be true when the original sentence is not. Again, decisions about this sort of thing depend heavily upon context and background assumptions. And there may be a distinction between what is *said* and what is conversationally *implied* in a given context. Perhaps what was said corresponds to the table, so that our translation is right, though there are certain assumptions typically made in conversation that go beyond. But we need not get into this. Our point is not that the ordinary 'and' *always* works like our operator $\wedge$; rather the point is that some (indeed, many) ordinary uses are rightly regarded as having the same table.[2] Again, we will ordinarily *assume* a context where 'and', 'but' and the like have tables that correspond to $\wedge$.

The operator which is most naturally associated with $\vee$ is '____ or ____'. In this case, there is room for caution from the start. Consider first a restaurant menu which says that you will get soup, or you will get salad, with your dinner. This is naturally understood as 'you will get soup or you will get salad' where the sentential operator is '____ or ____'. In this case, the table would seem to be,

(U)

____	or	____
T	***F***	T
T	***T***	F
F	***T***	T
F	***F***	F

The compound is true if you get soup, true if you get salad, but not if you get neither or both. None of our operators has this table.

But contrast this case with one where a professor promises either to give you an 'A' on a paper, or to give you very good comments so that you will know what went wrong. Suppose the professor gets excited about your paper, giving you both an 'A'

[2] The ability to make this point is an important byproduct of our having introduced the formal operators "as themselves." Where $\wedge$ and the like are introduced as *being* direct translations of ordinary operators, a natural reaction to cases of this sort — a reaction had even by some professional logicians and philosophers — is that "the table is wrong." But this is mistaken! $\wedge$ has its own significance, which may or may not agree with the shifting meaning of ordinary terms. The situation is no different than for translation across ordinary languages, where terms may or may not have uniform equivalents.

But now, one may feel a certain tension with our account of what it is for an operator to be truth functional — for there seem to be contexts where the truth value of sentences in the blanks does not determine the truth value of the whole, even for a purportedly truth functional operator like '____ and ____'. However, we want to distinguish different *senses* in which an operator may be used (or an *ambiguity*, as between a *bank* of a river, and a *bank* where you deposit money), so that when an operator is used with just one sense it has some definite truth function.

and comments. Presumably, she did not break her promise! That is, in this case, we seem to have, ‘I will give you an ‘A’ or I will give you comments’ with the table,

(V)

____	or	____
T	***T***	T
T	***T***	F
F	***T***	T
F	***F***	F

The professor breaks her word just in case she gives you a low grade without comments. This table is identical to the table for $\vee$. For another case, suppose you set out to buy a power saw, and say to your friend ‘I will go to Home Depot or I will go Lowes’. You go to Home Depot, do not find what you want, so go to Lowes and make your purchase. When your friend later asks where you went, and you say you went to both, he or she will not say you lied (!) when you said where you were going — for your statement required only that you would try at least one of those places.

The grading and shopping cases represent the so-called “inclusive” use of ‘or’ — including the case when both components are T; the menu uses the “exclusive” use of ‘or’ — excluding the case when both are T. Ordinarily, we will *assume* that ‘or’ is used in its inclusive sense, and so is translated directly by $\vee$.[3] Another operator that works this way is ‘____ unless ____’. Again, there are exclusive and inclusive senses — which you should be able to see by considering restaurant and grade examples as above. And again, we will ordinarily assume that the inclusive sense is intended. For the exclusive cases, we can generate the table by means of complex expressions. Thus, for example both $(\mathcal{P} \leftrightarrow \sim\mathcal{Q})$ and $[(\mathcal{P} \vee \mathcal{Q}) \wedge \sim(\mathcal{P} \wedge \mathcal{Q})]$ do the job. You should convince yourself that this is so.

Observe that ‘either ____ or ____’ has the same table as ‘____ or ____’; and ‘both ____ and ____’ the same as ‘____ and ____’. So one might think that ‘either’ and ‘both’ have no real role. They do however serve a sort of “bracketing” function: Consider ‘neither Bob likes Sue nor Sue likes Bob’. This is most naturally understood as, ‘it is not the case that either Bob likes Sue or Sue likes Bob’ with translation $\sim(B \vee S)$. Observe that this division is required: An attempt to parse it to ‘it is not the case that either Bob likes Sue or Sue like Bob’ results in the fragment ‘either Bob likes Sue’ in the blank for ‘it is not the case that ____’. There would be an ambiguity about the main operator if ‘either’ were missing; but with it there, the only way to keep complete sentences in the blanks is to make ‘it is not the case that ____’ the

[3] Again, there may be a distinction between what is *said* and what is conversationally *implied* in a given context. Perhaps what was said generally corresponds to the inclusive table, though many uses are against background assumptions which automatically exclude the case when both are T. But we need not get into this. It is enough that some uses are according to the inclusive table.

main operator. Similarly, 'not both Bob likes Sue and Sue likes Bob' comes to 'it is not the case that both Bob likes Sue and Sue likes Bob' with translation $\sim(B \wedge S)$. It is possible to make these points directly. Thus, for example, 'neither ____ nor ____' has the following table,

(W)

Neither	____ nor	____
F	T	T
F	T	F
F	F	T
T	F	F

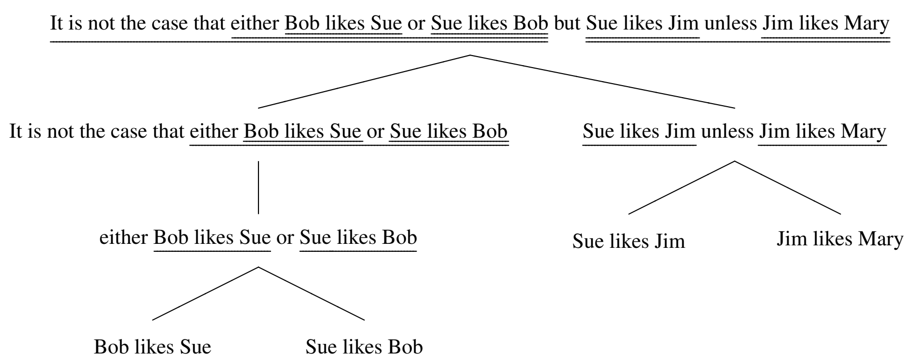
(X)

$\mathcal{P}$	$\mathcal{Q}$	$\sim$	($\mathcal{P}$	$\vee$	$\mathcal{Q}$)
T	T	***F***	T	T	F
T	F	***F***	T	T	T
F	T	***F***	F	T	F
F	F	***T***	F	F	T

From (W) 'neither Bob likes Sue nor Sue likes Bob' is true just when 'Bob likes Sue' and 'Sue likes Bob' are both false, and otherwise false. No operator of our formal language has a table which is T just when components are both F. Still, we may form complex expressions which work this way. So from (X), $\sim(\mathcal{P} \vee \mathcal{Q})$ has the same table. Another expression that works this way is $\sim\mathcal{P} \wedge \sim\mathcal{Q}$. Either would be a good translation, though one might be more natural than the other. Similarly both $\sim(\mathcal{P} \wedge \mathcal{Q})$ and $\sim\mathcal{P} \vee \sim\mathcal{Q}$ are a good translation for 'not both ____ and ____'.

And we continue to work with complex forms on trees. Thus, for example, consider 'Neither Bob likes Sue nor Sue likes Bob, but Sue likes Jim unless Jim likes Mary'. This is a mouthful, but we can deal with it in the usual way. The hard part, perhaps, is just exposing the operator structure.

(Y)



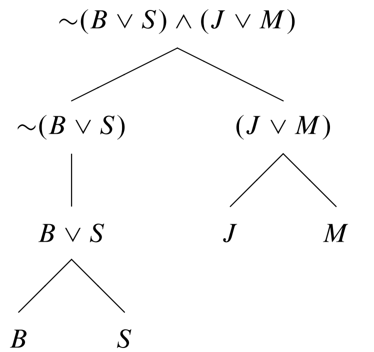
Given this, with what we have said above, generate the interpretation function and then the parallel tree as follows.

B: Bob likes Sue

S: Sue likes Bob

J: Sue likes Jim

M: Jim likes Mary



We have seen that '____ ∨ ____' is equivalent to '____ unless ____'; and that 'neither ____ nor ____' works like 'it is not the case that ____ or ____'. Given these, everything works as before. Again, the complex problem is rendered simple, if we attack it one operator at a time. Another option is $(\sim B \wedge \sim S) \wedge (J \vee M)$ with the alternate version of 'neither ____ nor ____'.

E5.8. Using the interpretation function below, produce parse trees and then parallel ones to complete the translation for each of the following.

B: Bob likes Sue

S: Sue likes Bob

B_1: Bob is cool

S_1: Sue is cool

a. Bob likes Sue.

b. Sue does not like Bob.

c. Bob likes Sue and Sue likes Bob.

d. Bob likes Sue or Sue likes Bob.

e. Bob likes Sue unless she is not cool.

f. Either Bob does not like Sue or Sue does not like Bob.

g. Neither Bob likes Sue, nor Sue likes Bob.

*h. Not both Bob and Sue are cool.

i. Bob and Sue are cool, and Bob likes Sue, but Sue does not like Bob.

j. Although neither Bob nor Sue are cool, either Bob likes Sue, or Sue likes Bob.

E5.9. Use our method to translate each of the following. That is, generate parse trees with an interpretation function for all the sentences, and then parallel trees to produce formal equivalents.[4]

a. Harry is not a muggle.

b. Neither Harry nor Hermione are muggles.

c. Either Harry's or Hermione's parents are muggles.

*d. Neither Harry, nor Ron, nor Hermione are muggles.

e. Not both Harry and Hermione have muggle parents.

f. The game of Quidditch continues unless the Snitch is caught.

*g. Although blatching and blagging are illegal in Quidditch, the woolongong shimmy is not.

h. Either the beater hits the bludger or you are not protected from it, and the bludger is a very heavy ball.

i. The Chudley Cannons are not the best Quidditch team ever, however they hope for the best.

j. Harry won the Quidditch cup in his 3rd year at Hogwarts, but not in his 1st, 2nd, 4th, or 5th.

5.2.5 If, Iff

The operator which is most naturally associated with $\rightarrow$ is 'if ____ then ____'. Consider some fellow, perhaps of less than sterling character, of whom we assert, 'If he loves her, then she is rich'. In this case, the table begins,

(Z)

If ____	then	____
T	***T***	T
T	***F***	F
F	?	T
F	***T***	F

[4] My source for the information on Quidditch is Kennilworthy Whisp (aka, J.K. Rowling), *Quidditch Through the Ages*, along with a daughter who is a rabid fan of all things Potter.

If 'He loves her' and 'She is rich' are both true, then what we said about him is true. If he loves her, but she is not rich, what we said was wrong. If he does not love her, and she is poor, then we are also fine, for all we said was that *if* he loves her, then she is rich. But what about the other case? Suppose he does not love her, but she is rich. There is a temptation to say that our conditional assertion is false. But do not give in! Notice: we did not say that he loves all the rich girls. All we said was that *if* he loves this particular girl, then she is rich. So the existence of rich girls he does not love does not undercut our claim. For another case, say you are trying to find the car he is driving and say 'If he is in his own car, then it is a Corvette.' That is, 'If he is in his own car then it is a Corvette'. You would be mistaken if he has traded his Corvette for a Yugo. But say the Corvette is in the shop and he is driving a loaner that also happens to be a Corvette. Then 'He is in his own car' is F and 'He is driving a Corvette' is T. Still, there is nothing wrong with your claim — *if* he is in his own car, then it is a Corvette. Given this, we are left with the completed table,

(AA)

If ___	then	___
T	***T***	T
T	***F***	F
F	***T***	T
F	***T***	F

which is identical to the table for $\rightarrow$. With L for 'He loves her' and R for 'She is rich', for 'If he loves her then she is rich' the natural translation is $(L \rightarrow R)$. Another case which works this way is He loves her only if she is rich. You should think through this as above. So far, perhaps, so good.

But the conditional calls for special comment. First, notice that the table shifts with the position of 'if'. Suppose he loves her if she is rich. Intuitively, this says the same as, 'If she is rich then he loves her'. This time, we are mistaken if she is rich and he does not love her. Thus, with the above table and assignments, we end up with translation $(R \rightarrow L)$. Notice that the order is switched around the arrow. We can make this point directly from the original claim.

(AB)

he loves her	if	she is rich
T	***T***	T
T	***T***	F
F	***F***	T
F	***T***	F

The claim is false just in the case where she is rich but he does not love her. The result is *not* the same as the table for $\rightarrow$. What we need is an expression that is F in the case when L is F and R is T, and otherwise T. We get just this with $(R \rightarrow L)$. Of course, this is just the same result as by intuitively reversing the operator into the regular 'If ____ then ____' form.

In the formal language, the *order* of the components is crucial. In a true material conditional, the truth of the antecedent guarantees the truth of the consequent. In ordinary language, this role is played, not by the order of the components, but by operator placement. In general, *if* by itself is an *antecedent* indicator; and *only if* is a *consequent* indicator. That is, we get,

(AC)	If $\mathcal{P}$ then $\mathcal{Q}$	$\Longrightarrow$	$(\mathcal{P} \rightarrow \mathcal{Q})$
	$\mathcal{P}$ if $\mathcal{Q}$	$\Longrightarrow$	$(\mathcal{Q} \rightarrow \mathcal{P})$
	$\mathcal{P}$ only if $\mathcal{Q}$	$\Longrightarrow$	$(\mathcal{P} \rightarrow \mathcal{Q})$
	only if $\mathcal{P}$, $\mathcal{Q}$	$\Longrightarrow$	$(\mathcal{Q} \rightarrow \mathcal{P})$

'If', taken alone, identifies what does the guaranteeing, and so the antecedent of our material conditional; 'only if' identifies what is guaranteed, and so the consequent.[5]

As we have just seen, the natural translation of '$\mathcal{P}$ if $\mathcal{Q}$' is $\mathcal{Q} \rightarrow \mathcal{P}$, and the translation of '$\mathcal{P}$ only if $\mathcal{Q}$' is $\mathcal{P} \rightarrow \mathcal{Q}$. Thus it should come as no surprise that the translation of '$\mathcal{P}$ if *and* only if $\mathcal{Q}$' is $(\mathcal{P} \rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \rightarrow \mathcal{P})$, where this is precisely what is abbreviated by $(\mathcal{P} \leftrightarrow \mathcal{Q})$. We can also make this point directly. Consider, 'he loves her if and only if she is rich'. The operator is truth functional, with the table,

	he loves her	if and only if	she is rich
(AD)	T	***T***	T
	T	***F***	F
	F	***F***	T
	F	***T***	F

It cannot be that he loves her and she is not rich, because he loves her *only if* she is rich; so the second row is F. And it cannot be that she is rich and he does not love her, because he loves her *if* she is rich; so the third row is F. The conditional is true just when both she is rich and he loves her, or neither. Another operator that works this way is '____ just in case ____'. You should convince yourself that this is so. Notice that 'if', 'only if', and 'if and only if' play very different roles for translation — you almost want to think of them as completely different words: *if*, *onlyif*, and *ifandonlyif*, each with their own distinctive logical role. Do not get the different roles confused!

For an example that puts some of this together, consider, 'She is rich if he loves her, if and only if he is a cad or very generous'. This comes to the following.

[5]It may feel natural to convert '$\mathcal{P}$ unless $\mathcal{Q}$' to '$\mathcal{P}$ if not $\mathcal{Q}$' and translate $(\sim\mathcal{Q} \rightarrow \mathcal{P})$. This is fine and, as is clear from the abbreviated form, equivalent to $(\mathcal{Q} \vee \mathcal{P})$. However, with the extra negation and concern about direction of the arrow, it is easy to get confused on this approach — so the simple wedge is less likely to go wrong.

Cause and Conditional

It is important that the material conditional does *not* directly indicate causal connection. Suppose we have sentences S: You strike the match, and L: The match will light. And consider,

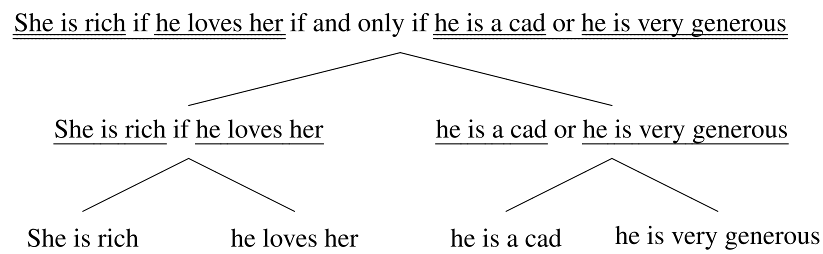
(i)	If you strike the match then it will light	$S \to L$
(ii)	The match will light only if you strike it	$L \to S$

with natural translations by our method on the right. Good. But, clearly the *cause* of the lighting is the striking. So the first arrow runs from cause to effect, and the second from effect to cause. Why? In (i) we represent the cause as *sufficient* for the effect: striking the match guarantees that it will light. In (ii) we represent the cause as *necessary* for the effect — the only way to get the match to light, is to strike it — so that the match's lighting guarantees that it was struck.

There may be a certain *tendency* to associate the ordinary 'if' and 'only if' with cause, so that we say, 'if $\mathcal{P}$ then $\mathcal{Q}$' when we think of $\mathcal{P}$ as a (sufficient) cause of $\mathcal{Q}$, and say '$\mathcal{P}$ only if $\mathcal{Q}$' when we think of $\mathcal{Q}$ as a (necessary) cause of $\mathcal{P}$. But causal direction is not reflected by the arrow, which comes out $(\mathcal{P} \to \mathcal{Q})$ either way. The material conditional indicates *guarantee*.

This point is important insofar as certain ordinary conditionals seem inextricably tied to causation. This is particularly the case with "subjunctive" conditionals (conditionals about what *would* have been). Suppose I was playing basketball and said, 'If I had played Kobe, I would have won' where this is, 'If it were the case that I played Kobe then it would have been the case that I won the game'. Intuitively, this is false, Kobe would wipe the floor with me. But contrast, 'If it were the case that I played Lassie then it would have been the case that I won the game'. Now, intuitively, this is true; Lassie has many talents but, presumably, basketball is not among them — and I could take her. But I have never played Kobe or Lassie, so both 'I played Kobe' and 'I played Lassie' are false. Thus the truth value of the whole conditional changes from false to true though the values of sentences in the blanks remain the same; and 'If it were the case that ____ then it would have been the case that ____' is not even truth functional. Subjunctive conditionals do offer a sort of guarantee, but the guarantee is for situations alternate to the way things actually are. So actual truth values do not determine the truth of the conditional.

Conditionals other than the material conditional are a central theme of Priest, *Non-Classical Logics*. As usual, we simply assume that 'if' and 'only if' are used in their truth functional sense, and so are given a good translation by $\to$.

(AE) 

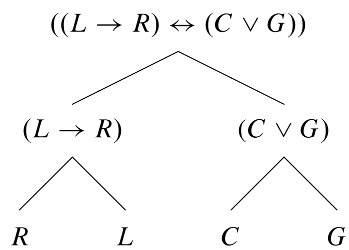
We begin by assigning sentence letters to the simple sentences at the bottom. Then the parallel tree is constructed as follows.

R: She is rich

L: He loves her

C: He is a cad

G: He is very generous



Observe that she is rich if he loves her is equivalent to $(L \rightarrow R)$, not the other way around. Then the wedge translates '____ or ____', and the main operator has the same table as $\leftrightarrow$.

Notice again that our procedure for translating, one operator or part at a time, lets us translate even where the original is so complex that it is difficult to comprehend. The method forces us to capture all available truth functional structure, and the translation is thus good insofar as given the specified interpretation function, the method makes the formal sentence true at just the consistent stories where the original is true. It does this because the formal and informal sentences *work* the same way. Eventually, you want to be able to work translations without the trees! (And maybe you have already begun to do so.) In fact, it will be helpful to generate them from the *top down*, rather than from the bottom up, building the translation operator-by-operator as you take the sentence apart from the main operator. But, of course, the result should be the same no matter how you do it.

From definition AR on p. 5 an argument is some sentences, one of which (the conclusion) is taken to be supported by the remaining sentences (the premises). In some courses on logic or critical reasoning, one might spend a great deal of time learning to identify premises and conclusions in ordinary discourse. However, we have taken this much as given, representing arguments in *standard form*, with premises listed as complete sentences above a line, and the conclusion under. Thus, for example,

	If you strike the match, then it will light
(AF)	The match will not light
	You did not strike the match

is a simple argument of the sort we might have encountered in chapter 1. To translate the argument, we produce a translation for the premises and conclusion, retaining the "standard-form" structure. Thus as in the discussion of causation on p. 108, we might end up with an interpretation function and translation as below,

S: You strike the match

L: The match will light

$$\begin{array}{l} S \to L \\ \sim L \\ \hline \sim S \end{array}$$

The result is an object to which we can apply our semantic and derivation methods in a straightforward way.

And this is what we have been after: If a formal argument is sententially valid, then the corresponding ordinary argument must be logically valid. For some good formal translation of its premises and conclusion, suppose an argument is sententially valid; then by SV there is *no* interpretation on which the premises are true and the conclusion is false; so there is no *intended* interpretation on which the premises are true and the conclusion is false; but given a good translation, by CG, the ordinary-language premises and conclusion have the same truth values at any consistent story as formal expressions on the corresponding intended interpretation; so no *consistent story* has the premises true and the conclusion false; so by LV the original argument is logically valid. We will make this point again, in some detail, in **??**. For now, notice that our formal methods, derivations and truth tables, apply to arguments of arbitrary complexity. So we are in a position to demonstrate validity for arguments that would have set us on our heels in chapter 1. With this in mind, consider again the butler case (B) that we began with from p. 2. The demonstration that the argument is logically valid is entirely straightforward, by a good translation and then truth tables to demonstrate semantic validity. (It remains for **??** to show how *derivations* matter for semantic validity.)

E5.10. Using the interpretation function below, produce parse trees and then parallel ones to complete the translation for each of the following.

L: Lassie barks

T: Timmy is in trouble

P: Pa will help

H: Lassie is healthy

a. If Timmy is in trouble, then Lassie barks.

b. Timmy is in trouble if Lassie barks.

c. Lassie barks only if Timmy is in trouble.

d. If Timmy is in trouble and Lassie barks, then Pa will help.

*e. If Timmy is in trouble, then if Lassie barks Pa will help.

f. If Pa will help only if Lassie barks, then Pa will help if and only if Timmy is in trouble.

g. Pa will help if Lassie barks, just in case Lassie barks only if Timmy is in trouble.

h. If Timmy is in trouble and Pa does not help, then Lassie is not healthy or does not bark.

*i. If Timmy is in trouble, then either Lassie is not healthy or if Lassie barks then Pa will help.

j. If Lassie neither barks nor is healthy, then Timmy is in trouble if Pa will not help.

E5.11. Use our method, with or without parse trees, to produce a translation, including interpretation function for the following.

a. If animals feel pain, then animals have intrinsic value.

b. Animals have intrinsic value only if they feel pain.

c. Although animals feel pain, vegetarianism is not right.

d. Animals do not have intrinsic value unless vegetarianism is not right.

e. Vegetarianism is not right only if animals do not feel pain or do not have intrinsic value.

f. If you think animals feel pain, then vegetarianism is right.

*g. If you think animals do not feel pain, then vegetarianism is not right.

h. If animals feel pain, then if animals have intrinsic value if they feel pain, then animals have intrinsic value.

*i. Vegetarianism is right only if both animals feel pain, and animals have intrinsic value just in case they feel pain; but it is not the case that animals have intrinsic value just in case they feel pain.

j. If animals do not feel pain if and only if you think animals do not feel pain, but you do think animals feel pain, then you do not think that animals feel pain.

E5.12. For each of the following arguments: (i) Produce a good translation, including interpretation function and translations for the premises and conclusion. Then (ii) use truth tables to determine whether the argument is sententially valid.

*a. Our car will not run unless it has gasoline
Our car has gasoline
———
Our car will run

b. If Bill is president, then Hillary is first lady
Hillary is not first lady
———
Bill is not president

c. Snow is white and snow is not white
———
Dogs can fly

d. If Mustard murdered Boddy, then it happened in the library.
The weapon was the pipe if and only if it did not happen in the library, and the weapon was not the pipe only if Mustard murdered him
———
Mustard murdered Boddy

e. There is evil
If god is good, there is no evil unless he has an excuse for allowing it.
If god is omnipotent, then he does not have an excuse for allowing evil.
———
God is not both good and omnipotent.

E5.13. For each of the arguments in E512 that is sententially valid, produce a derivation to show that it is valid in *AD*.

E5.14. Use translation and truth tables to show that the butler argument (B) from p. 2 is semantically valid.

E5.15. For each of the following concepts, explain in an essay of about two pages, so that (high-school age) Hannah could understand. In your essay, you should (i) identify the objects to which the concept applies, (ii) give and explain the definition, and give and explicate examples of your own construction (iii) where the concept applies, and (iv) where it does not. Your essay should exhibit an understanding of methods from the text.

a. Good translations.

b. Truth functional operators

c. Parse trees, interpretation functions and parallel trees

5.3 Quantificational

Chapter 6

Natural Deduction

Natural deductions systems are so-called because their rules formalize patterns of reasoning that occur in relatively ordinary "natural" contexts. Thus, initially at least, the rules of natural deduction systems are easier to motivate than the axioms and rules of axiomatic systems. By itself, this is sufficient to give natural deduction a special interest. As we shall see, natural deduction is also susceptible to proof *strategies* in a way that (primitive) axiomatic systems are not. If you have had another course in formal logic, you have probably been exposed to natural deduction. So, again, it may seem important to bring what we have done into contact with what you have encountered in other contexts. After some general remarks about natural deduction, we turn to the sentential and quantificational components of our system *ND*, and finally to an expanded system, *ND*+.

6.1 General

I begin this section with a few general remarks about derivation systems and derivation rules. We will then turn to some background notions for the particular rules of our official natural derivation systems.[1]

6.1.1 Derivations as Games

In their essential nature, derivations are defined in terms of form. Both axiomatic and natural derivations can be seen as a kind of game — with the aim of getting from a starting point to a goal by rules. Taken as games, there is no immediate or

[1]Parts of this section are reminiscent of 3.1 and, especially if you skipped over that section, you may want to look over it now as additional background.

obvious connection between derivations and semantic validity or truth. This point may have been particularly vivid with respect to axiomatic systems. In the case of natural derivations, the systems are driven by *rules* rather than axioms, and the rules may "make sense" in a way that axioms do not. Still, we can introduce natural derivations purely in their nature as games. Thus, for example, consider a system $N1$ with the following rules.

$N1$ R1 $\dfrac{\mathcal{P} \rightarrow \mathcal{Q}, \mathcal{P}}{\mathcal{Q}}$ R2 $\dfrac{\mathcal{P} \vee \mathcal{Q}}{\mathcal{Q}}$ R3 $\dfrac{\mathcal{P} \wedge \mathcal{Q}}{\mathcal{P}}$ R4 $\dfrac{\mathcal{P}}{\mathcal{P} \vee \mathcal{Q}}$

In this system, R1: given formulas of the form $\mathcal{P} \rightarrow \mathcal{Q}$ and $\mathcal{P}$, one may move to $\mathcal{Q}$; R2: given a formula of the form $\mathcal{P} \vee \mathcal{Q}$, one may move to $\mathcal{Q}$; R3: given a formula of the form $\mathcal{P} \wedge \mathcal{Q}$, one may move to $\mathcal{P}$; and R4: given a formula $\mathcal{P}$ one may move to $\mathcal{P} \vee \mathcal{Q}$ for any $\mathcal{Q}$. For now, at least, the game is played as follows: One begins with some starting formulas and a goal. The starting formulas are like "cards" in your hand. One then applies the rules to obtain more formulas, to which the rules may be applied again and again. You win if you eventually obtain the goal formula. Each application of a rule is *independent* of the ones before — so all that matters for a given move is whether formulas are of the requisite forms; it does not matter what was $\mathcal{P}$ or what was $\mathcal{Q}$ in a previous application of the rules.

Let us consider some examples. At this stage, do not worry about strategy, about why we do what we do, as much as about how the rules work and the way the game is played. A game always begins with starting premises at the top, and goal on the bottom.

(A)

1.	$A \rightarrow (B \wedge C)$	P(remise)
2.	A	P(remise)
	$B \vee D$	(goal)

The formulas on lines (1) and (2) are of the form $\mathcal{P} \rightarrow \mathcal{Q}$ and $\mathcal{P}$, where $\mathcal{P}$ maps to A and $\mathcal{Q}$ to $(B \wedge C)$; so we are in a position to apply rule R1 to get the $\mathcal{Q}$.

1.	$A \rightarrow (B \wedge C)$	P(remise)
2.	A	P(remise)
3.	$B \wedge C$	1,2 R1
	$B \vee D$	(goal)

The justification for our move — the way the rules apply — is listed on the right; in this case, we use the formulas on lines (1) and (2) according to rule R1 to get $B \wedge C$;

so that is indicated by the notation. Now, $B \wedge C$ is of the form $\mathcal{P} \wedge \mathcal{Q}$. So we can apply R3 to it in order to obtain the $\mathcal{P}$, namely B.

1.	$A \to (B \wedge C)$	P(remise)
2.	A	P(remise)
3.	$B \wedge C$	1,2 R1
4.	B	3 R3
	$B \vee D$	(goal)

Notice that one application of a rule is independent of another. It does not matter what formula was $\mathcal{P}$ or $\mathcal{Q}$ in a previous move, for evaluation of this one. Finally, where $\mathcal{P}$ is B, $B \vee D$ is of the form $\mathcal{P} \vee \mathcal{Q}$. So we can apply R4 to get the final result.

1.	$A \to (B \wedge C)$	P(remise)
2.	A	P(remise)
3.	$B \wedge C$	1,2 R1
4.	B	3 R3
5.	$B \vee D$	4 R4 Win!

Notice that R4 leaves the $\mathcal{Q}$ unrestricted: Given some $\mathcal{P}$, we can move to $\mathcal{P} \vee \mathcal{Q}$ for *any* $\mathcal{Q}$. Since we reached the goal from the starting sentences, we win! In this simple derivation system, any line of a successful derivation is a premise, or justified from lines before by the rules.

Here are a couple more examples, this time of completed derivations.

(B)

1.	$A \wedge C$	P
2.	$(A \vee B) \to D$	P
3.	A	1 R3
4.	$A \vee B$	3 R4
5.	D	2,4 R1
6.	$D \vee (R \to S)$	5 R4 Win!

$A \wedge C$ is of the form $\mathcal{P} \wedge \mathcal{Q}$. So we can apply R3 to obtain the $\mathcal{P}$, in this case A. Then where $\mathcal{P}$ is A, we use R4 to add on a B to get $A \vee B$. $(A \vee B) \to D$ and $A \vee B$ are of the form $\mathcal{P} \to \mathcal{Q}$ and $\mathcal{P}$; so we apply R1 to get the $\mathcal{Q}$, that is D. Finally, where D is $\mathcal{P}$, $D \vee (R \to S)$ is of the form $\mathcal{P} \vee \mathcal{Q}$; so we apply R4 to get the final result. Notice again that the $\mathcal{Q}$ may be any formula whatsoever.

Here is another example.

(C)

1.	$(A \wedge B) \wedge D$	P
2.	$(A \wedge B) \rightarrow C$	P
3.	$A \rightarrow (C \rightarrow (B \wedge D))$	P
4.	$A \wedge B$	1 R3
5.	C	2,4 R1
6.	A	4 R3
7.	$C \rightarrow (B \wedge D)$	3,6 R1
8.	$B \wedge D$	7,5 R1
9.	B	8 R3 Win!

You should be able to follow the steps. In this case, we use $A \wedge B$ on line (4) twice; once as part of an application of R1 to get C, and again in an application of R3 to get the A. Once you have a formula in your "hand" you can use it as many times and whatever way the rules will allow. Also, the order in which we worked might have been different. Thus, for example, we might have obtained A on line (5) and then C after. You win if you get to the goal by the rules; how you get there is up to you. Finally, it is tempting to think we could get B from, say, $A \wedge B$ on line (4). We will able to do this in our official system. But the rules we have so far do not let us do so. R3 lets us move just to the left conjunct of a formula of the form $\mathcal{P} \wedge \mathcal{Q}$.

When there is a way to get from the premises of some argument to its conclusion by the rules of derivation system *N*, the premises *prove* the conclusion in system *N*. In this case, where Γ (Gamma) is the set of premises, and $\mathcal{P}$ the conclusion we write $\Gamma \vdash_N \mathcal{P}$. If $\Gamma \vdash_N \mathcal{P}$ the argument is *valid* in derivation system *N*. Notice the distinction between this "single turnstile" $\vdash$ and the double turnstile $\vDash$ associated with semantic validity. As usual, if $\mathcal{Q}_1 \ldots \mathcal{Q}_n$ are the members of Γ, we sometimes write $\mathcal{Q}_1 \ldots \mathcal{Q}_n \vdash_N \mathcal{P}$ in place of $\Gamma \vdash_N \mathcal{P}$. If Γ has no members then, listing all the members of Γ individually, we simply write $\vdash_N \mathcal{P}$. In this case, $\mathcal{P}$ is a *theorem* of derivation system *N*.

One can imagine setting up many different rule sets, and so many different games of this kind. In the end, we want our game to serve a specific purpose. That is, we want to use the game in the identification of valid arguments. In order for our games to be an indicator of validity, we would like it to be the case that $\Gamma \vdash_N \mathcal{P}$ iff $\Gamma \vDash \mathcal{P}$, that Γ *proves* $\mathcal{P}$ iff Γ *entails* $\mathcal{P}$. In **??** we will show that our official derivation games have this property.

For now, we can at least see how this might be: Roughly, we impose the following condition on rules: we require of our rules that *the inputs always semantically entail the outputs*. Then if some premises are true, and we make a move to a formula, the formula we move to must be true; and if the formulas in our "hand" are all true, and we add some formula by another move, the formula we add must be true; and so forth for each formula we add until we get to the goal, which will have to be true as well. So if the premises are true, the goal must be true as well. We will have much more to say about this later!

For now, notice that our rules R1, R3 and R4 each meet the proposed requirement on rules, but R2 does not.

(D)

		R1			R2		R3		R4	
$\mathscr{P}$	$\mathscr{Q}$	$\mathscr{P} \rightarrow \mathscr{Q}$	$\mathscr{P}$	/ $\mathscr{Q}$	$\mathscr{P} \vee \mathscr{Q}$	/ $\mathscr{Q}$	$\mathscr{P} \wedge \mathscr{Q}$	/ $\mathscr{P}$	$\mathscr{P}$	/ $\mathscr{P} \vee \mathscr{Q}$
T	T	***T***	***T***	***T***	***T***	***T***	***T***	***T***	***T***	***T***
T	F	***F***	***T***	***F***	***T***	***F***	***F***	***T***	***T***	***T***
F	T	***T***	***F***	***T***	***T***	***T***	***F***	***F***	***F***	***T***
F	F	***T***	***F***	***F***	***F***	***F***	***F***	***F***	***F***	***F***

R1, R3 and R4 have no row where the input(s) are T and the output is F. But for R2, the second row has input T and output F. So R2 does not meet our condition. This does not mean that one cannot construct a *game* with R2 as a part. Rather, the point is that R2 will not help us accomplish what we want to accomplish with our games. As we demonstrate in **??**, so long as rules meet the condition, a win in the game always corresponds to an argument that is semantically valid. Thus, for example, derivation (C), in which R2 does not appear, corresponds to the result that $(A \wedge B) \wedge D, (A \wedge B) \rightarrow C, A \rightarrow (C \rightarrow (B \wedge D)) \vDash_s B$.

(E)

A	B	C	D	$(A \wedge B)$	$\wedge D$	$(A \wedge B)$	$\rightarrow C$	$A \rightarrow$	$(C \rightarrow$	$(B \wedge D))$	/ B
T	T	T	T	T	***T***	T	***T***	***T***	T	T	***T***
T	T	T	F	T	***F***	T	***T***	***F***	F	F	***T***
T	T	F	T	T	***T***	T	***F***	***T***	T	T	***T***
T	T	F	F	T	***F***	T	***F***	***T***	T	F	***T***
T	F	T	T	F	***F***	F	***T***	***F***	F	F	***F***
T	F	T	F	F	***F***	F	***T***	***F***	F	F	***F***
T	F	F	T	F	***F***	F	***T***	***T***	T	F	***F***
T	F	F	F	F	***F***	F	***T***	***T***	T	F	***F***
F	T	T	T	F	***F***	F	***T***	***T***	T	T	***T***
F	T	T	F	F	***F***	F	***T***	***T***	F	F	***T***
F	T	F	T	F	***F***	F	***T***	***T***	T	T	***T***
F	T	F	F	F	***F***	F	***T***	***T***	T	F	***T***
F	F	T	T	F	***F***	F	***T***	***T***	F	F	***F***
F	F	T	F	F	***F***	F	***T***	***T***	F	F	***F***
F	F	F	T	F	***F***	F	***T***	***T***	T	F	***F***
F	F	F	F	F	***F***	F	***T***	***T***	T	F	***F***

There is no row where the premises are T and the conclusion is F. As the number of rows goes up, we may decide that the games are dramatically easier to complete than the tables. And derivations are particularly important in the quantificational case, where we have not yet been able to demonstrate semantic validity at all.

E6.1. Show that each of the following is valid in $N1$. Complete (a) - (d) using just rules R1, R3 and R4. You will need an application of R2 for (e).

*a. $(A \wedge B) \wedge C \vdash_{N1} A$

b. $(A \wedge B) \wedge C, A \rightarrow (B \wedge C) \vdash_{N1} B$

c. $(A \wedge B) \rightarrow (B \wedge A), A \wedge B \vdash_{N1} B \vee A$

d. $R, [R \vee (S \vee T)] \rightarrow S \vdash_{N1} S \vee T$

e. $A \vdash_{N1} A \rightarrow C$

*E6.2. (i) For each of the arguments in E6.1, use a truth table to decide if the argument is sententially valid. (ii) To what do you attribute the fact that a win in $N1$ is not a sure indicator of semantic validity?

6.1.2 Auxiliary Assumptions

So far, our derivations have had the following form,

(F)

a.	$\mathcal{A}$	P(remise)
	$\vdots$	
b.	$\mathcal{B}$	P(remise)
	$\vdots$	
c.	$\mathcal{G}$	(goal)

We have some premise(s) at the top, and a conclusion at the bottom. The premises are against a line which indicates the range or *scope* over which the premises apply. In each case, the line extends from the premises to the conclusion, indicating that the conclusion is derived from them. It is always our aim to derive the conclusion under the scope of the premises alone. But our official derivation system will allow appeal to certain *auxiliary* assumptions in addition to premises. Any such assumption comes with a scope line of its own — indicating the range over which *it* applies. Thus, for example, derivations might be structured as follows.

(G)

a.	$\mathcal{A}$	P(remise)
b.	$\mathcal{B}$	P(remise)
c.	$\mathcal{C}$	A(ssumption)
d.		
e.	$\mathcal{G}$	(goal)

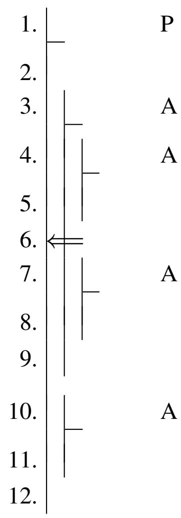
(H)

a.	$\mathcal{A}$	P(remise)
b.	$\mathcal{B}$	P(remise)
c.	$\mathcal{C}$	A(ssumption)
d.	$\mathcal{D}$	A(ssumption)
e.		
f.		
g.	$\mathcal{G}$	(goal)

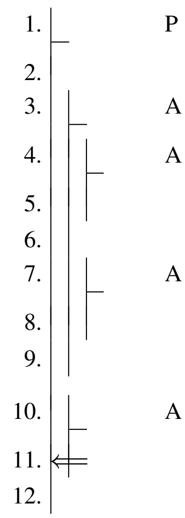
In each, there are premises $\mathcal{A}$ through $\mathcal{B}$ at the top and goal $\mathcal{G}$ at the bottom. As indicated by the main leftmost scope line, the premises apply throughout the derivations, and the goal is derived under them. In case (G), there is an additional assumption at (c). As indicated by its scope line, that assumption applies from (c) - (d). In (H), there are a pair of additional assumptions. As indicated by the associated scope lines, the first applies over (c) - (f), and the second over (d) - (e). We will say that an auxiliary assumption, together with the formulas that fall under its scope, is a *subderivation*. Thus (G) has a subderivation on from (c) - (d). (H) has a pair of subderivations, one on (c) - (f), and another on (d) - (e). A derivation or subderivation may *include* various other subderivations. Any subderivation begins with an auxiliary assumption. In general we *cite* a subderivation by listing the line number on which it begins, then a dash, and the line number on which its scope line ends.

In contexts without auxiliary assumptions, we have been able freely to appeal to any formula already in our "hand." Where there are auxiliary assumptions, however, we may appeal only to *accessible* subderivations and formulas. A formula is *accessible* at a given stage when it is obtained under assumptions all of which continue to apply. In practice, what this means is that for justification of a formula at line number i we can appeal only to formulas which appear immediately against scope lines extending as far as i. Thus, for example, with the scope structure as in (I) below, in the justification of line (6),

(I)



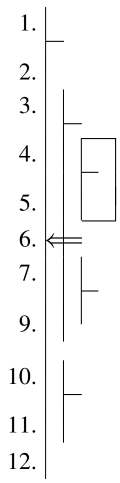
(J)



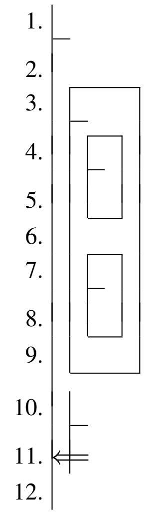
we could appeal only to formulas at (1), (2) and (3), for these are the only ones immediately against scope lines extending as far as (6). To see this, notice that scope lines extending as far as (6), are ones cut by the arrow at (6). Formulas at (4) and (5) are not against a line extending that far. Similarly, as indicated by the arrow in (J), for the justification of (11), we could appeal only to formulas at (1), (2), and (10). Formulas at other line numbers are not immediately against scope lines extending as far as (11). The accessible formulas are ones derived under assumptions all of which continue to apply.

It may be helpful to think of a completed subderivation as a sort of "box." So long as you are under the scope of an assumption, the box is open and you can "see" the formulas under its scope. However, once you exit from an assumption, the box is closed, and the formulas inside are no longer available.

(I′)



(J′)



Thus, again, in (I′) the formulas at (4) - (5) are locked away so that the only accessible lines are (1) - (3). Similarly, at line (11) of (J′) all of (3) - (9) is unavailable.

Our aim is always to obtain the goal against the leftmost scope line — under the scope of the premises alone — and if the only formulas accessible for its justification are also against the leftmost scope line, it may appear mysterious why we would ever introduce auxiliary assumptions and subderivations at all. What is the point of auxiliary assumptions, if formulas under their scope are inaccessible for justification for the formula we want? The answer is that, though the formulas inside a box are unavailable *the box* may still be useful. Certain of our rules will appeal to entire subderivations (to the boxes), rather than to the formulas in them. A subderivation is *accessible* at a given stage when *it* is obtained under assumptions all of which continue to apply. In practice, what this means is that for a formula at line i, we can appeal to a box (to a subderivation) only if *it* (its scope line) is against a line which extends down to i.

Thus at line (6) of (I′), we would not be able to appeal to the formulas on lines (4) and (5) — they are inside the closed box. However, we *would* be able to appeal to the *box* on lines (4) - (5), for *it* is against a scope line cut by the arrow. Similarly, at line (11) of (J′) we are not able to appeal to formulas on any of the lines (3) - (9), for they are inside the closed boxes. Similarly, we cannot appeal to the *boxes* on (4) - (5) or (7) - (8) for they are locked inside the larger box. However, we can appeal to the larger subderivation on (3) - (9) insofar as it is against a line cut by the arrow. Observe that one can appeal to a box only after it is closed – so, for example, at (11) of (J′) there is not (yet) a closed box at (10) - (11) and so no available subderivation to which one may appeal.

Putting this together, at (12) we can appeal to the subderivations at (3) - (9) and (10) - (11); the ones at (4) - (5) and (7) - (8) remain inaccessible. The justification for line (12) might therefore appeal to the formulas on lines (1) and (2) or to the subderivations on lines (3) - (9) and (10) - (11). Again line (12) does not have access to the *formulas* inside the subderivations from lines (3) - (9) and (10) - (11). So the subderivations are accessible even where the formulas inside them are not, and there may be a point to the subderivations even where the formulas *inside* the subderivation are inaccessible.

Definitions for Auxiliary Assumptions

SD An auxiliary assumption, together with the formulas that fall under its scope, is a *subderivation*.

FA A formula is *accessible* at a given stage when it is obtained under assumptions all of which continue to apply.

SA A subderivation is *accessible* at a given stage when it (as a whole) is obtained under assumptions all of which continue to apply.

In practice, what this means is that for justification of a formula at line i we can appeal to another formula only if it is immediately against a scope line extending as far as i.

And in practice, for justification of a formula at line i, we can appeal to a subderivation only if its whole *scope line* is itself immediately against a scope line extending as far as i.

All this will become more concrete as we turn now to the rules of our official system *ND*. At this stage, we set aside the rules of our preliminary system $N1$ and begin again from scratch. We can reinforce the point about accessibility of *formulas* by introducing the first, and simplest, rule of our official system. If a formula $\mathcal{P}$ appears on an accessible line *a* of a derivation, we may repeat it by the rule *reiteration*, with justification *a* R.

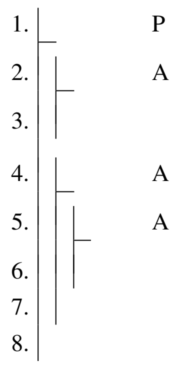
R

a.	$\mathcal{P}$	
	$\mathcal{P}$	a R

It should be obvious why reiteration satisfies our basic condition on rules. If $\mathcal{P}$ is true, *of course* $\mathcal{P}$ is true. So this rule could never lead from a formula that is true, to one that is not. Observe, though, that the line *a* must be *accessible*. If in (I) the assumption at line (3) were a formula $\mathcal{P}$, then we could conclude $\mathcal{P}$ with justification 3 R at lines (5), (6), (8) or (9). We could not obtain $\mathcal{P}$ with the same justification at

(11) or (12) without violating the rule, because (3) is not accessible for justification of (11) or (12). You should be clear about why this is so.

*E6.3. Consider a derivation with the following structure.



For each of the lines (3), (6), (7) and (8) which lines are accessible? which subderivations (if any) are accessible? That is, complete the following table.

	accessible lines	accessible subderivations
line 3		
line 6		
line 7		
line 8		

*E6.4. Suppose in a derivation with structure as in E6.3 we have obtained a formula $\mathcal{A}$ on line (3). (i) On what lines would we be allowed to conclude $\mathcal{A}$ by 3 R? Suppose there is a formula $\mathcal{B}$ on line (4). (ii) On what lines would we be allowed to conclude $\mathcal{B}$ by 4 R? Hint: this is just a question about accessibility, asking where it is possible to use lines (3) and (4).

6.2 Sentential

Our system *N*1 set up the basic idea of derivations as games. We begin presentation of our official natural deduction system *ND* with rules whose application is just to sentential forms — to forms involving $\sim$, and $\rightarrow$ (and so to $\wedge$, $\vee$, and $\leftrightarrow$). Though the only operators in the forms are sentential, the forms may apply to expressions in either a sentential language like $\mathcal{L}_s$, or a quantificational one like $\mathcal{L}_q$. For the most part, though, we simply focus on $\mathcal{L}_s$. In a derivation, each formula is either a premise, an auxiliary assumption, or is justified by the rules. As we will see, auxiliary assumptions are always introduced in conjunction with an *exit strategy*. In addition to reiteration, the sentential part of *ND* includes two rules for each of the five sentential operators — for a total of eleven rules. For each of the operators, there is an 'I' or *introduction* rule, and an 'E' or *exploitation* rule.[2] As we will see, this division helps structure the way we approach derivations: To generate a formula with main operator $\star$, you will typically use the corresponding introduction rule. To make use of a formula with main operator $\star$, you will typically employ the exploitation rule for that operator.

6.2.1 $\to$ and $\wedge$

Let us start with the I- and E-rules for $\to$ and $\wedge$. We have already seen the exploitation rule for $\to$. It is R1 of system *N*1. If formulas $\mathcal{P} \to \mathcal{Q}$ and $\mathcal{P}$ and appear on accessible lines *a* and *b* of a derivation, we may conclude $\mathcal{Q}$ with justification *a,b* $\to$E.

$\to$E

a.	$\mathcal{P} \to \mathcal{Q}$	
b.	$\mathcal{P}$	
	$\mathcal{Q}$	a,b $\to$E

Intuitively, if it is true that *if* $\mathcal{P}$ *then* $\mathcal{Q}$, and it is true that $\mathcal{P}$, then $\mathcal{Q}$ must be true as well. And, on table (D) we saw that if both $\mathcal{P} \to \mathcal{Q}$ and $\mathcal{P}$ are true, then $\mathcal{Q}$ is true. Notice that we *do not* somehow get the $\mathcal{P}$ from $\mathcal{P} \to \mathcal{Q}$. Rather, we exploit $\mathcal{P} \to \mathcal{Q}$ when, given that $\mathcal{P}$ also is true, we use $\mathcal{P}$ together with $\mathcal{P} \to \mathcal{Q}$ to conclude $\mathcal{Q}$. So this rule requires two input "cards." The $\mathcal{P} \to \mathcal{Q}$ card sits idle without a $\mathcal{P}$ to activate it. The order in which $\mathcal{P} \to \mathcal{Q}$ and $\mathcal{P}$ appear does not matter so long as they are both accessible. However, you should cite them in the standard order — line for the conditional first, then the antecedent. As in the axiomatic system from chapter 3, this rule is sometimes called *modus ponens*.

Here is an example. We show, $L, L \to (A \wedge K), (A \wedge K) \to (L \to P) \vdash_{ND} P$.

(K)

1.	L	P
2.	$L \to (A \wedge K)$	P
3.	$(A \wedge K) \to (L \to P)$	P
4.	$A \wedge K$	2,1 $\to$E
5.	$L \to P$	3,4 $\to$E
6.	P	5,1 $\to$E

[2] I- and E-rules are often called *introduction* and *elimination* rules. This can lead to confusion as E-rules do not necessarily eliminate anything. The above, which is becoming more common, is more clear.

$L \to (A \wedge K)$ and L and are of the form $\mathscr{P} \to \mathscr{Q}$ and $\mathscr{P}$ where L is the $\mathscr{P}$ and $A \wedge K$ is $\mathscr{Q}$. So we use them to conclude $A \wedge K$ by →E on (4). But then $(A \wedge K) \to (L \to P)$ and $A \wedge K$ are of the form $\mathscr{P} \to \mathscr{Q}$ and $\mathscr{P}$, so we use them to conclude $\mathscr{Q}$, in this case, $L \to P$, on line (5). Finally $L \to P$ and L are of the form $\mathscr{P} \to \mathscr{Q}$ and $\mathscr{P}$, and we use them to conclude P on (6). Notice that,

(L)

1.	$(A \to B) \wedge C$	P	
2.	A	P	
3.	B	1,2 →E	Mistake!

misapplies the rule. $(A \to B) \wedge C$ is not of the form $\mathscr{P} \to \mathscr{Q}$ — the main operator being $\wedge$, so that the formula is of the form $\mathscr{P} \wedge \mathscr{Q}$. The rule →E applies just to formulas with main operator →. If we want to use $(A \to B) \wedge C$ with A to conclude B, we would first have to isolate $A \to B$ on a line of its own. We might have done this in $N1$. But there is no rule for this (yet) in *ND*!

→I is our first rule that requires a subderivation. Once we understand this rule, the rest are mere variations on a theme. →I takes as its input an entire subderivation. Given an accessible subderivation which begins with assumption $\mathscr{P}$ on line *a* and ends with $\mathscr{Q}$ against the assumption's scope line at *b*, one may conclude $\mathscr{P} \to \mathscr{Q}$ with justification *a-b* →I.

→I

a.	$\mathscr{P}$	A ($\mathscr{Q}$, →I)
b.	$\mathscr{Q}$	
	$\mathscr{P} \to \mathscr{Q}$	a-b →I

or

a.	$\mathscr{P}$	A (g, →I)
b.	$\mathscr{Q}$	
	$\mathscr{P} \to \mathscr{Q}$	a-b →I

Note that the auxiliary assumption comes with a stated *exit strategy*: In this case the exit strategy includes the *formula* $\mathscr{Q}$ with which the subderivation is to end, and an indication of the rule (→I) by which exit is to be made. We might write out the entire formula inside the parentheses as on the left. In practice, however, this is tedious, and it is easier just to write the formula at the bottom of the scope line where we will need it in the end. Thus in the parentheses on the right '*g*' is a simple *pointer* to the goal formula at the end of the scope line. Note that the pointer is empty unless there is a formula to which it points, and *the exit strategy therefore is not complete unless the goal formula is stated*. In this case, the strategy includes the pointer to the goal formula, along with the indication of the rule (→I) by which exit is to be made. Again, at the time we make the assumption, we write the $\mathscr{Q}$ down as part of the strategy for exiting the subderivation. But this does not mean the $\mathscr{Q}$ is justified! The $\mathscr{Q}$ is rather introduced as a new goal. Notice also that the justification *a-b* →I does not refer to the *formulas* on lines *a* and *b*. These are inaccessible. Rather, the justification appeals to the subderivation which begins on line *a* and ends on line *b* —

where this subderivation is accessible even though the formulas in it are not. So there is a difference between the comma and the hyphen, as they appear in justifications.

For this rule, we assume the antecedent, reach the consequent, and conclude to the conditional by →I. Intuitively, if an assumption $\mathcal{P}$ leads to $\mathcal{Q}$ then we know that *if* $\mathcal{P}$ *then* $\mathcal{Q}$. On truth tables, if there is a sententially valid argument from some other premises together with assumption $\mathcal{P}$ to conclusion $\mathcal{Q}$, then there is no row where those other premises are true and the assumption $\mathcal{P}$ is true but $\mathcal{Q}$ is false — but this is just to say that there is no row where the other premises are true and $\mathcal{P} \to \mathcal{Q}$ is false. We will have much more to say about this in **??**.

For an example, suppose we are confronted with the following.

(M)

1.	$A \to B$	P
2.	$B \to C$	P
	$A \to C$	

In general, we use an introduction rule to *produce* some formula — typically one already given as a goal. →I generates $\mathcal{P} \to \mathcal{Q}$ given a subderivation that starts with the $\mathcal{P}$ and ends with the $\mathcal{Q}$. Thus to reach $A \to C$, we need a subderivation that starts with A and ends with C. So we set up to reach $A \to C$ with the assumption A and an exit strategy to produce $A \to C$ by →I. For this we set the consequent C as a subgoal.

1.	$A \to B$	P
2.	$B \to C$	P
3.	A	A (g, →I)
	C	
	$A \to C$	

Again, we have not yet reached C or $A \to C$. Rather, we have assumed A and set C as a subgoal, with the strategy of terminating our subderivation by an application of →I. This much is stated in the exit strategy. As it happens, C is easy to get.

1.	$A \to B$	P
2.	$B \to C$	P
3.	A	A (g, →I)
4.	B	1,3 →E
5.	C	2,4 →E
	$A \to C$	

Having reached C, and so completed the subderivation, we are in a position to execute our exit strategy and conclude $A \to C$ by →I.

1.	$A \to B$	P
2.	$B \to C$	P
3.	$\quad$ A	A (g, →I)
4.	$\quad$ B	1,3 →E
5.	$\quad$ C	2,4 →E
6.	$A \to C$	3-5 →I

We appeal to the subderivation that starts with the assumption of the antecedent, and reaches the consequent. Notice that the →I setup is driven, not by available premises and assumptions, but by where we want to get. We will say something more systematic about strategy once we have introduced all the rules. But here is the fundamental idea: *think goal directedly*. We begin with $A \to C$ as a goal. Our idea for producing it leads to C as a new goal. And the new goal is relatively easy to obtain.

Here is another example, one that should illustrate the above point about strategy, as well as the rule. Say we want to show $A \vdash_{ND} B \to (C \to A)$.

(N)

1.	A	P
	$B \to (C \to A)$	

Forget about the premise! Since the goal is of the form $\mathcal{P} \to \mathcal{Q}$, we set up to get it by →I.

1.	A	P
2.	$\quad$ B	A (g, →I)
	$\quad$ $C \to A$	
	$B \to (C \to A)$	

We need a subderivation that starts with the antecedent, and ends with the consequent. So we assume the antecedent, and set the consequent as a new goal. In this case, the new goal $C \to A$ has main operator →, so we set up again to reach *it* by →I.

1.	A	P
2.	$\quad B$	A (g, $\rightarrow$I)
3.	$\quad\quad C$	A (g, $\rightarrow$I)
	$\quad\quad A$	
	$\quad C \rightarrow A$	
	$B \rightarrow (C \rightarrow A)$	

The pointer g in an exit strategy points to the goal formula at the bottom of its scope line. Thus g for assumption B at (2) points to $C \rightarrow A$ at the bottom of its line, and g for assumption C at (3) points to A at the bottom of *its* line. Again, for the conditional, we assume the antecedent, and set the consequent as a new goal. And this last goal is particularly easy to reach. It follows immediately by reiteration from (1). Then it is a simple matter of executing the exit strategies with which our auxiliary assumptions were introduced.

1.	A	P
2.	$\quad B$	A (g, $\rightarrow$I)
3.	$\quad\quad C$	A (g, $\rightarrow$I)
4.	$\quad\quad A$	1 R
5.	$\quad C \rightarrow A$	3-4 $\rightarrow$I
6.	$B \rightarrow (C \rightarrow A)$	2-5 $\rightarrow$I

The subderivation which begins on (3) and ends on (4) begins with the antecedent and ends with the consequent of $C \rightarrow A$. So we conclude $C \rightarrow A$ on (5) by 3-4 $\rightarrow$I. The subderivation which begins on (2) and ends at (5) begins with the antecedent and ends with the consequent of $B \rightarrow (C \rightarrow A)$. So we reach $B \rightarrow (C \rightarrow A)$ on (6) by 2-5 $\rightarrow$I. Notice again how our overall reasoning is driven by the goals, rather than the premises and assumptions. It is sometimes difficult to motivate strategy when derivations are short and relatively easy. But this sort of thinking will stand you in good stead as problems get more difficult!

Given what we have done, the E- and I- rules for $\wedge$ are completely straightforward. If $\mathcal{P} \wedge \mathcal{Q}$ appears on some accessible line a of a derivation, then you may move to the $\mathcal{P}$, or to the $\mathcal{Q}$ with justification a $\wedge$E.

$\wedge$E	a.	$\mathcal{P} \wedge \mathcal{Q}$		a.	$\mathcal{P} \wedge \mathcal{Q}$	
		$\mathcal{P}$	a $\wedge$E		$\mathcal{Q}$	a $\wedge$E

Either qualifies as an instance of the rule. The left-hand case was R3 from $N1$. Intuitively, $\wedge$E should be clear. If $\mathcal{P}$ *and* $\mathcal{Q}$ is true, then $\mathcal{P}$ is true. And if $\mathcal{P}$ *and* $\mathcal{Q}$ is true, then $\mathcal{Q}$ is true. We saw a table for the left-hand case in (D). The other is similar. The $\wedge$ introduction rule is equally straightforward. If $\mathcal{P}$ and $\mathcal{Q}$ appear on accessible lines *a* and *b* of a derivation, then you may move to $\mathcal{P} \wedge \mathcal{Q}$ with justification *a,b* $\wedge$I.

$\wedge$I

a.	$\mathcal{P}$	
b.	$\mathcal{Q}$	
	$\mathcal{P} \wedge \mathcal{Q}$	a,b $\wedge$I

The order in which $\mathcal{P}$ and $\mathcal{Q}$ appear is irrelevant, though you should cite them in the specified order, line for the left conjunct first, and then for the right. If $\mathcal{P}$ is true and $\mathcal{Q}$ is true, then $\mathcal{P}$ *and* $\mathcal{Q}$ is true. Similarly, on a table, any line with both $\mathcal{P}$ and $\mathcal{Q}$ true has $\mathcal{P} \wedge \mathcal{Q}$ true.

Here is a simple example, demonstrating the *associativity* of conjunction.

(O)

1.	$A \wedge (B \wedge C)$	P
2.	A	1 $\wedge$E
3.	$B \wedge C$	1 $\wedge$E
4.	B	3 $\wedge$E
5.	C	3 $\wedge$E
6.	$A \wedge B$	2,4 $\wedge$I
7.	$(A \wedge B) \wedge C$	6,5 $\wedge$I

Notice that we could not get the B alone or the C alone without first isolating $B \wedge C$ on (3). As before, our rules apply just to the *main* operator. In effect, we take apart the premise with the E-rule, and put the conclusion together with the I-rule. Of course, as with $\rightarrow$I and $\rightarrow$E, rules for other operators do not always let us get to the parts and put them together in this simple and symmetric way.

Words to the wise:

- A common mistake made by beginning students is to assimilate other rules to $\wedge$E and $\wedge$I — moving, say, from $\mathcal{P} \to \mathcal{Q}$ alone to $\mathcal{P}$ or $\mathcal{Q}$, or from $\mathcal{P}$ and $\mathcal{Q}$ to $\mathcal{P} \to \mathcal{Q}$. *Do not forget what you have learned! Do not make this mistake!* The $\wedge$ rules are particularly easy. But each operator has its own special character. Thus $\to$E requires two "cards" to play. And $\to$I takes a subderivation as input.

- Another common mistake is to assume a formula $\mathcal{P}$ merely because it would be nice to have access to $\mathcal{P}$. *Do not make this mistake!* An assumption always comes with an exit strategy, and is useful only for application of the exit rule. At this stage, then, the *only* reason to assume $\mathcal{P}$ is to produce a formula of the sort $\mathcal{P} \to \mathcal{Q}$ by $\to$I.

A final example brings together all of the rules so far (except R).

(P)

1.	$A \to C$	P
2.	$\quad$ $A \wedge B$	A (g, $\to$I)
3.	$\quad$ A	2 $\wedge$E
4.	$\quad$ C	1,3 $\to$E
5.	$\quad$ B	2 $\wedge$E
6.	$\quad$ $B \wedge C$	5,4 $\wedge$I
7.	$(A \wedge B) \to (B \wedge C)$	2-6 $\to$I

We set up to obtain the overall goal by $\to$I. This generates $B \wedge C$ as a subgoal. We get $B \wedge C$ by getting the B and the C. Here is our guiding idea for strategy (which may now seem obvious): As you focus on a goal, to generate a formula with main operator $\star$, consider producing it by $\star$I. Thus, if the main operator of a goal or subgoal is $\to$, consider producing the formula by $\to$I; if the main operator of a goal is $\wedge$, consider producing it by $\wedge$I. This much should be sufficient for you to approach the following exercises. As you do the derivations, it is good simply to leave plenty of space on the page for your derivation as you state goal formulas, and let there be blank lines if room remains.[3]

[3] Typing on a computer, it is easy to push lines down if you need more room. It is not so easy with pencil and paper, and worse with pen! If you decide to type, most word processors have a symbol font, with the capability of assigning symbols to particular keys. Assigning keys is far more efficient than finding characters over and over in menus.

E6.5. Complete the following derivations by filling in justifications for each line. Hint: it may be convenient to xerox the problems, and fill in your answers directly on the copy.

a.

1. $(A \wedge B) \rightarrow C$
2. $B \wedge A$
3. B
4. A
5. $A \wedge B$
6. C

b.

1. $(R \rightarrow L) \wedge [(S \vee R) \rightarrow (T \leftrightarrow K)]$
2. $(R \rightarrow L) \rightarrow (S \vee R)$
3. $R \rightarrow L$
4. $S \vee R$
5. $(S \vee R) \rightarrow (T \leftrightarrow K)$
6. $T \leftrightarrow K$

c.

1. B
2. $(A \rightarrow B) \rightarrow (B \rightarrow (L \wedge S))$
3. $\quad | \; A$
4. $\quad | \; B$
5. $A \rightarrow B$
6. $B \rightarrow (L \wedge S)$
7. $L \wedge S$
8. S
9. L
10. $S \wedge L$

d.

1. $A \wedge B$
2. $\quad | \; C$
3. $\quad | \; A$
4. $\quad | \; A \wedge C$
5. $C \rightarrow (A \wedge C)$
6. $\quad | \; C$
7. $\quad | \; B$
8. $\quad | \; B \wedge C$
9. $C \rightarrow (B \wedge C)$
10. $[C \rightarrow (A \wedge C)] \wedge [C \rightarrow (B \wedge C)]$

e.

1. $(A \wedge S) \to C$
2. $\quad A$
3. $\quad\quad S$
4. $\quad\quad A \wedge S$
5. $\quad\quad C$
6. $\quad S \to C$
7. $A \to (S \to C)$

E6.6. The following are not legitimate *ND* derivations. In each case, explain why.

*a.

1. $(A \wedge B) \wedge (C \to B)$ P
2. A 1 $\wedge$E

b.

1. $(A \wedge B) \wedge (C \to A)$ P
2. C P
3. A 1,2 $\to$E

c.

1. $(R \wedge S) \wedge (C \to A)$ P
2. $C \to A$ 1 $\wedge$E
3. A 2 $\to$E

d.

1. $A \to B$ P
2. $\quad A \wedge C$ A (g, $\to$I)
3. $\quad A$ 2 $\wedge$E
4. B 1,3 $\to$E

e.

1. $A \to B$ P
2. $\quad A \wedge C$ A (g, $\to$I)
3. $\quad A$ 2 $\wedge$E
4. $\quad B$ 1,3 $\to$E
5. $\quad C$ 2 $\wedge$E
6. $\quad A \wedge C$ 3,5 $\wedge$I

Hint: For this problem, think carefully about the exit strategy and the scope lines. Do we have the conclusion where we want it?

E6.7. Provide derivations to show each of the following.

a. $A \wedge B \vdash_{ND} B \wedge A$

*b. $A \wedge B, B \rightarrow C \vdash_{ND} C$

c. $A \wedge (A \rightarrow (A \wedge B)) \vdash_{ND} B$

d. $A \wedge B, B \rightarrow (C \wedge D) \vdash_{ND} A \wedge D$

*e. $A \rightarrow (A \rightarrow B) \vdash_{ND} A \rightarrow B$

f. $A, (A \wedge B) \rightarrow (C \wedge D) \vdash_{ND} B \rightarrow C$

g. $C \rightarrow A, C \rightarrow (A \rightarrow B) \vdash_{ND} C \rightarrow (A \wedge B)$

*h. $A \rightarrow B, B \rightarrow C \vdash_{ND} (A \wedge K) \rightarrow C$

i. $A \rightarrow B \vdash_{ND} (A \wedge C) \rightarrow (B \wedge C)$

j. $D \wedge E, (D \rightarrow F) \wedge (E \rightarrow G) \vdash_{ND} F \wedge G$

k. $O \rightarrow B, B \rightarrow S, S \rightarrow L \vdash_{ND} O \rightarrow L$

*l. $A \rightarrow B \vdash_{ND} (C \rightarrow A) \rightarrow (C \rightarrow B)$

m. $A \rightarrow (B \rightarrow C) \vdash_{ND} B \rightarrow (A \rightarrow C)$

n. $A \rightarrow (B \rightarrow C), D \rightarrow B \vdash_{ND} A \rightarrow (D \rightarrow C)$

o. $A \rightarrow B \vdash_{ND} A \rightarrow (C \rightarrow B)$

6.2.2 $\sim$ and $\vee$

Now let us consider the I- and E-rules for $\sim$ and $\vee$. The two rules for $\sim$ are quite similar to one another. Each appeals to a single subderivation. For $\sim$I, given an accessible subderivation which begins with assumption $\mathcal{P}$ on line a, and ends with a formula of the form $\mathcal{Q} \wedge \sim\mathcal{Q}$ against its scope line on line b, one may conclude $\sim\mathcal{P}$ by a-b $\sim$I. For $\sim$E, given an accessible subderivation which begins with assumption $\sim\mathcal{P}$ on line a, and ends with a formula of the form $\mathcal{Q} \wedge \sim\mathcal{Q}$ against its scope line on line b, one may conclude $\mathcal{P}$ by a-b $\sim$E.

$\sim$I

a.	$\mathcal{P}$	A (c, $\sim$I)
b.	$\mathcal{Q} \wedge \sim\mathcal{Q}$	
	$\sim\mathcal{P}$	a-b $\sim$I

$\sim$E

a.	$\sim\mathcal{P}$	A (c, $\sim$E)
b.	$\mathcal{Q} \wedge \sim\mathcal{Q}$	
	$\mathcal{P}$	a-b $\sim$E

$\sim$I introduces an expression with main operator tilde, adding tilde to the assumption $\mathcal{P}$. $\sim$E exploits the assumption $\sim\mathcal{P}$, with a result that takes the tilde off. For these rules, the formula $\mathcal{Q}$ may be *any* formula, so long as $\sim\mathcal{Q}$ is *it* with a tilde in front. Because $\mathcal{Q}$ may be any formula, when we declare our exit strategy for the assumption, we might have no particular goal formula in mind. So, where g always points to a formula written at the bottom of a scope line, c is not a pointer to any particular formula. Rather, when we declare our exit strategy, we merely indicate our intent to obtain some contradiction, and then to exit by $\sim$I or $\sim$E.

Intuitively, if an assumption leads to a result that is false, the assumption is wrong. So if the assumption $\mathcal{P}$ leads to $\mathcal{Q} \wedge \sim\mathcal{Q}$, then $\sim\mathcal{P}$; and if the assumption $\sim\mathcal{P}$ leads to $\mathcal{Q} \wedge \sim\mathcal{Q}$, then $\mathcal{P}$. On tables, there can be no row where $\mathcal{Q} \wedge \sim\mathcal{Q}$ is true; so if every row where some premises together with assumption $\mathcal{P}$ are true would have to make $\mathcal{Q} \wedge \sim\mathcal{Q}$ true, then there can be no row where those other premises are true and $\mathcal{P}$ is true — so any row where the other premises are true is one where $\mathcal{P}$ is false, and $\sim\mathcal{P}$ is therefore true. Similarly when the assumption is $\sim\mathcal{P}$, any row where the other premises are true has to be one where $\sim\mathcal{P}$ is false, so that $\mathcal{P}$ is true. Again, we will have much more to say about this reasoning in **??**.

Here are some examples of these rules. Notice that, again, we introduce subderivations with the overall goal in mind.

(Q)

1.	$A \rightarrow B$		P
2.	$A \rightarrow \sim B$		P
3.		A	A (c, $\sim$I)
4.		B	1,3 $\rightarrow$E
5.		$\sim B$	2,3 $\rightarrow$E
6.		$B \wedge \sim B$	4,5 $\wedge$I
7.	$\sim A$		3-6, $\sim$I

We begin with the goal of obtaining $\sim A$. The natural way to obtain this is by $\sim$I. So we set up a subderivation with that in mind. Since the goal is $\sim A$, we begin with A, and go for a contradiction. In this case, the contradiction is easy to obtain, by a couple applications of $\rightarrow$E and then $\wedge$I.

Here is another case that may be more interesting.

(R)

1.	$\sim A$	P
2.	$B \to A$	P
3.	$\quad L \wedge B$	A (c, $\sim$I)
4.	$\quad B$	3 $\wedge$E
5.	$\quad A$	2,4 $\to$E
6.	$\quad A \wedge \sim A$	5,1 $\wedge$I
7.	$\sim(L \wedge B)$	3-6 $\sim$I

This time, the original goal is $\sim(L \wedge B)$. It is of the form $\sim\mathcal{P}$, so we set up to obtain it with a subderivation that begins with the $\mathcal{P}$, that is, $L \wedge B$. In this case, the contradiction is $A \wedge \sim A$. Once we have the contradiction, we simply apply our exit strategy.

A simplification. Let $\bot$ (bottom) abbreviate an arbitrary contradiction — say $Z \wedge \sim Z$. Adopt a rule $\bot$I as on the left below,

$\bot$I

a.	$\mathcal{Q}$	
b.	$\sim\mathcal{Q}$	
	$\bot$	a,b $\bot$I

(S)

1.	$\mathcal{Q}$	
2.	$\sim\mathcal{Q}$	
3.	$\quad \sim\bot$	A (c $\sim$E)
4.	$\quad \mathcal{Q} \wedge \sim\mathcal{Q}$	1,2 $\wedge$I
5.	$\bot$	3-4 $\sim$E

Given $\mathcal{Q}$ and $\sim\mathcal{Q}$ on accessible lines, we move directly to $\bot$ by $\bot$I. This is an example of a *derived* rule. For given $\mathcal{Q}$ and $\sim\mathcal{Q}$, we can always derive $\bot$ as in (S) on the right. Thus we allow ourselves to shortcut the routine by introducing $\bot$I as a derived rule. We will see examples of additional derived rules in **??**. For now, the important thing is that since $\bot$ abbreviates $Z \wedge \sim Z$ we *operate* on $\bot$ as we might operate on $Z \wedge \sim Z$. Thus, for example, we might derive $\bot$ from Z and $\sim Z$ by $\wedge$I; or use $\wedge$E to conclude Z or $\sim Z$ from $\bot$. Especially, given this abbreviation, our $\sim$I and $\sim$E rules appear in forms,

$\sim$I

a.	$\quad \mathcal{P}$	A (c, $\sim$I)
b.	$\quad \bot$	
	$\sim\mathcal{P}$	a-b $\sim$I

$\sim$E

a.	$\quad \sim\mathcal{P}$	A (c, $\sim$E)
b.	$\quad \bot$	
	$\mathcal{P}$	a-b $\sim$E

Since $\bot$ is (abbreviates) the sentence $Z \wedge \sim Z$, the subderivations for $\sim$I and $\sim$E are appropriately concluded with $\bot$. Observe that with $\bot$ at the bottom the $\sim$I and $\sim$E rules have a particular goal sentence, very much like $\to$I. However, the $\mathcal{Q}$ and $\sim\mathcal{Q}$ required to obtain $\bot$ by $\bot$I are the same as would be required for $\mathcal{Q} \wedge \sim\mathcal{Q}$ on the original form of the rules. For this reason, we declare our exit strategy with a

c rather than g any time the goal is $\bot$. At one level, this simplification is a mere notational convenience: having obtained $\mathcal{Q}$ and $\sim\mathcal{Q}$, we move to $\bot$, instead of writing out the complex conjunction $\mathcal{Q} \wedge \sim\mathcal{Q}$. However, there are contexts where it will be convenient to have a *particular* contradiction as goal. Thus this is the standard form in which we use these rules.

Here is an example of the rules in this form, this time for $\sim$E.

(T)

1.	$\sim\sim A$	P
2.	$\quad \sim A$	A (c, $\sim$E)
3.	$\quad \bot$	2,1 $\bot$I
4.	A	2-3 $\sim$E

It is no surprise that we can derive A from $\sim\sim A$! This is how to do it in *ND*. Again, do not begin by thinking about the premise. The goal is A, and we can get it with a subderivation that starts with $\sim A$, by a $\sim$E exit strategy. In this case the $\mathcal{Q}$ and $\sim\mathcal{Q}$ for $\bot$I are $\sim A$ and $\sim\sim A$ — that is $\sim A$ with a tilde in front of it. Though very often (at least in the beginning) an atomic and its negation will do for your contradiction, $\mathcal{Q}$ and $\sim\mathcal{Q}$ need not be simple. Observe that $\sim$E is a strange and powerful rule: Though an E-rule, effectively it can be used in pursuit of any goal whatsoever — to obtain formula $\mathcal{P}$ by $\sim$E, all one has to do is obtain a contradiction from the assumption of $\mathcal{P}$ with a tilde in front. As in this last example (T), $\sim$E is particularly useful when the goal is an atomic formula, and thus without a main operator, so that there is no straightforward way for regular introduction rules to apply. In this way, it plays the role of a sort of "backdoor" introduction rule.

The $\vee$I and and $\vee$E rules apply methods we have already seen. For $\vee$I, given an accessible formula $\mathcal{P}$ on line a, one may move to either $\mathcal{P} \vee \mathcal{Q}$ or to $\mathcal{Q} \vee \mathcal{P}$ for any formula $\mathcal{Q}$, with justification a $\vee$I.

$\vee$I

a.	$\mathcal{P}$		a.	$\mathcal{P}$	
	$\mathcal{P} \vee \mathcal{Q}$	a $\vee$I		$\mathcal{Q} \vee \mathcal{P}$	a $\vee$I

The left-hand case was R4 from $N1$. Also, we saw an intuitive version of this rule as *addition* on p. 28. Table (D) exhibits the left-hand case. And the other side should be clear as well: Any row of a table where $\mathcal{P}$ is true has both $\mathcal{P} \vee \mathcal{Q}$ and $\mathcal{Q} \vee \mathcal{P}$ true.

Here is a simple example.

(U)

1.	P	P
2.	$(P \vee Q) \rightarrow R$	P
3.	$P \vee Q$	1 $\vee$I
4.	R	2,3 $\rightarrow$E

It is easy to get R once we have $P \vee Q$. And we build $P \vee Q$ directly from the P. Note that we could have done the derivation as well if (2) had been, say, $(P \vee [K \wedge (L \leftrightarrow T)]) \rightarrow R$ and we used $\vee$I to add $[K \wedge (L \leftrightarrow T)]$ to the P all at once.

The inputs to $\vee$E are a formula of the form $\mathcal{P} \vee \mathcal{Q}$ and *two* subderivations. Given an accessible formula of the form $\mathcal{P} \vee \mathcal{Q}$ on line a, with an accessible subderivation beginning with assumption $\mathcal{P}$ on line b and ending with conclusion $\mathcal{C}$ against its scope line at c, and an accessible subderivation beginning with assumption $\mathcal{Q}$ on line d and ending with conclusion $\mathcal{C}$ against its scope line at e, one may conclude $\mathcal{C}$ with justification *a,b-c,d-e* $\vee$E.

$\vee$E	a.	$\mathcal{P} \vee \mathcal{Q}$	
	b.	$\quad \mathcal{P}$	A (g, a$\vee$E)
	c.	$\quad \mathcal{C}$	
	d.	$\quad \mathcal{Q}$	A (g, a$\vee$E)
	e.	$\quad \mathcal{C}$	
		$\mathcal{C}$	a,b-c,d-e $\vee$E

Given a disjunction $\mathcal{P} \vee \mathcal{Q}$, one subderivation begins with $\mathcal{P}$, and the other with $\mathcal{Q}$; both concluding with $\mathcal{C}$. This time our exit strategy includes markers for the new subgoals, along with a notation that we exit by appeal to the disjunction on line a and $\vee$E. Intuitively, if we know it is one or the other, and *either* leads to some conclusion, then the conclusion must be true. Here is an example a student gave me near graduation time: She and her mother were shopping for a graduation dress. They narrowed it down to dress A or dress B. Dress A was expensive, and if they bought it, her mother would be mad. But dress B was ugly and if they bought it the student would complain and her mother would be mad. Conclusion: her mother would be mad — and this without knowing which dress they were going to buy! On a truth table, if rows where $\mathcal{P}$ is true have $\mathcal{C}$ true, and rows where $\mathcal{Q}$ is true have $\mathcal{C}$ true, then any row with $\mathcal{P} \vee \mathcal{Q}$ true must have $\mathcal{C}$ true as well.

Here are a couple of examples. The first is straightforward, and illustrates both the $\vee$I and $\vee$E rules.

(V)

1.	$A \vee B$	P
2.	$A \to C$	P
3.	$\quad A$	A (g, 1$\vee$E)
4.	$\quad C$	2,3 $\to$E
5.	$\quad B \vee C$	4 $\vee$I
6.	$\quad B$	A (g, 1$\vee$E)
7.	$\quad B \vee C$	6 $\vee$I
8.	$B \vee C$	1,3-5,6-7 $\vee$E

We have the disjunction $A \vee B$ as premise, and original goal $B \vee C$. And we set up to obtain the goal by $\vee$E. For this, one subderivation starts with A and ends with $B \vee C$, and the other starts with B and ends with $B \vee C$. As it happens, these subderivations are easy to complete.

Very often, beginning students resist using $\vee$E — no doubt because it is relatively messy. *But this is a mistake — $\vee$E is your friend*! In fact, with this rule, we have a case where it pays to look at the premises for general strategy. Again, we will have more to say later. But if you have a premise or accessible line of the form $\mathcal{P} \vee \mathcal{Q}$, you should go for your goal, whatever it is, by $\vee$E. Here is why: As you go for the goal in the first subderivation, you have whatever premises were accessible before, *plus* $\mathcal{P}$; and as you go for the goal in the second subderivation, you have whatever premises were accessible before *plus* $\mathcal{Q}$. So you can only be better off in your quest to reach the goal. In many cases where a premise has main operator $\vee$, there is no way to complete the derivation except by $\vee$E. The above example (V) is a case in point.

Here is a relatively messy example, which should help you be sure you understand the $\vee$ rules. It illustrates the *associativity* of disjunction.

(W)

1.	$A \vee (B \vee C)$	P
2.	$\quad A$	A (g, 1$\vee$E)
3.	$\quad A \vee B$	2 $\vee$I
4.	$\quad (A \vee B) \vee C$	3 $\vee$I
5.	$\quad B \vee C$	A (g, 1$\vee$E)
6.	$\quad\quad B$	A (g, 5$\vee$E)
7.	$\quad\quad A \vee B$	6 $\vee$I
8.	$\quad\quad (A \vee B) \vee C$	7 $\vee$I
9.	$\quad\quad C$	A (g, 5$\vee$E)
10.	$\quad\quad (A \vee B) \vee C$	9 $\vee$I
11.	$\quad (A \vee B) \vee C$	5,6-8,9-10 $\vee$E
12.	$(A \vee B) \vee C$	1,2-4,5-11 $\vee$E

The premise has main operator $\vee$. So we set up to obtain the goal by $\vee$E. This gives us subderivations starting with A and $B \vee C$, each with $(A \vee B) \vee C$ as goal. The first is easy to complete by a couple instances of $\vee$I. But the assumption of the second, $B \vee C$ has main operator $\vee$. So we set up to obtain *its* goal by $\vee$E. This gives us subderivations starting with B and C, each again having $(A \vee B) \vee C$ as goal. Again, these are easy to complete by application of $\vee$I. The final result follows by the planned applications of $\vee$E. If you have been able to follow this case, you are doing well!

E6.8. Complete the following derivations by filling in justifications for each line.

a.

1.	$\sim B$
2.	$(\sim A \vee C) \to (B \wedge C)$
3.	$\quad \sim A$
4.	$\quad \sim A \vee C$
5.	$\quad B \wedge C$
6.	$\quad B$
7.	$\quad \bot$
8.	A

b.

1. | R
2. | $\sim(S \vee T)$
3. | | $R \to S$
4. | | S
5. | | $S \vee T$
6. | | $\bot$
7. | $\sim(R \to S)$

c.

1. | $(R \wedge S) \vee (K \wedge L)$
2. | | $R \wedge S$
3. | | R
4. | | S
5. | | $S \wedge R$
6. | | $(S \wedge R) \vee (L \wedge K)$
7. | | $K \wedge L$
8. | | K
9. | | L
10. | | $L \wedge K$
11. | | $(S \wedge R) \vee (L \wedge K)$
12. | $(S \wedge R) \vee (L \wedge K)$

d.

1. | $A \vee B$
2. | | A
3. | | | $A \to B$
4. | | | B
5. | | $(A \to B) \to B$
6. | | B
7. | | | $A \to B$
8. | | | B
9. | | $(A \to B) \to B$
10. | $(A \to B) \to B$

e.

$$
\begin{array}{rl}
1. & \mid \sim B \\
2. & \mid \sim A \rightarrow (A \vee B) \\
 & \mid\!\!\text{---} \\
3. & \mid \mid \sim A \\
 & \mid \mid\!\!\text{---} \\
4. & \mid \mid A \vee B \\
5. & \mid \mid \mid A \\
 & \mid \mid \mid\!\!\text{---} \\
6. & \mid \mid \mid A \\
7. & \mid \mid \mid B \\
 & \mid \mid \mid\!\!\text{---} \\
8. & \mid \mid \mid \mid \sim A \\
 & \mid \mid \mid \mid\!\!\text{---} \\
9. & \mid \mid \mid \mid \bot \\
10. & \mid \mid \mid A \\
11. & \mid \mid A \\
12. & \mid \mid \bot \\
13. & \mid A
\end{array}
$$

E6.9. The following are not legitimate *ND* derivations. In each case, explain why.

a.

$$
\begin{array}{rll}
1. & \mid A \vee B & \text{P} \\
 & \mid\!\!\text{---} & \\
2. & \mid B & 1\ \vee\text{E}
\end{array}
$$

b.

$$
\begin{array}{rll}
1. & \mid \sim A & \text{P} \\
2. & \mid B \rightarrow A & \text{P} \\
 & \mid\!\!\text{---} & \\
3. & \mid \mid B & \text{A } (c, \sim\text{I}) \\
 & \mid \mid\!\!\text{---} & \\
4. & \mid \mid A & 2,3\ \rightarrow\text{E} \\
5. & \mid \sim B & 3\text{-}4\ \sim\text{I}
\end{array}
$$

*c.

$$
\begin{array}{rll}
1. & \mid W & \text{P} \\
 & \mid\!\!\text{---} & \\
2. & \mid \mid R & \text{A } (c, \sim\text{I}) \\
 & \mid \mid\!\!\text{---} & \\
3. & \mid \mid \mid \sim W & \text{A } (c, \sim\text{I}) \\
 & \mid \mid \mid\!\!\text{---} & \\
4. & \mid \mid \mid \bot & 1,3\ \bot\text{I} \\
5. & \mid \sim R & 2\text{-}4\ \sim\text{I}
\end{array}
$$

d.

1.	$A \vee B$		P
2.		A	A (g, 1$\vee$E)
3.		A	2 R
4.		B	A (g, 1$\vee$E)
5.		A	3 R
6.	A		1,2-3,4-5 $\vee$E

e.

1.	$A \vee B$			P
2.		A		A (g, 1$\vee$E)
3.		A		2 R
4.		A		A (c, $\sim$I)
5.			B	A (g, 1$\vee$E)
6.			A	4 R
7.	A			1,2-3,5-6 $\vee$E

E6.10. Produce derivations to show each of the following.

a. $\sim A \vdash_{ND} \sim(A \wedge B)$

b. $A \vdash_{ND} \sim\sim A$

*c. $\sim A \rightarrow B, \sim B \vdash_{ND} A$

d. $A \rightarrow B \vdash_{ND} \sim(A \wedge \sim B)$

e. $\sim A \rightarrow B, B \rightarrow A \vdash_{ND} A$

f. $A \wedge B \vdash_{ND} (R \leftrightarrow S) \vee B$

*g. $A \vee (A \wedge B) \vdash_{ND} A$

h. $S, (B \vee C) \rightarrow \sim S \vdash_{ND} \sim B$

i. $A \vee B, A \rightarrow B, B \rightarrow A \vdash_{ND} A \wedge B$

j. $A \rightarrow B, (B \vee C) \rightarrow D, D \rightarrow \sim A \vdash_{ND} \sim A$

k. $A \vee B \vdash_{ND} B \vee A$

*l. $A \rightarrow \sim B \vdash_{ND} B \rightarrow \sim A$

m. $(A \wedge B) \rightarrow \sim A \vdash_{ND} A \rightarrow \sim B$

n. $A \vee \sim\sim B \vdash_{ND} A \vee B$

o. $A \vee B, \sim B \vdash_{ND} A$

6.2.3 $\leftrightarrow$

We complete our presentation of rules for the sentential part of *ND* with the rules $\leftrightarrow$E and $\leftrightarrow$I. Given that $\mathcal{P} \leftrightarrow \mathcal{Q}$ abbreviates the same as $(\mathcal{P} \rightarrow \mathcal{Q}) \wedge (\mathcal{Q} \rightarrow \mathcal{P})$, it is not surprising that rules for $\leftrightarrow$ work like ones for arrow, but going two ways. For $\leftrightarrow$E, if formulas $\mathcal{P} \leftrightarrow \mathcal{Q}$ and $\mathcal{P}$ appear on accessible lines *a* and *b* of a derivation, we may conclude $\mathcal{Q}$ with justification *a,b* $\leftrightarrow$E; and similarly but in the other direction, if formulas $\mathcal{P} \leftrightarrow \mathcal{Q}$ and $\mathcal{Q}$ appear on accessible lines *a* and *b* of a derivation, we may conclude $\mathcal{P}$ with justification *a,b* $\leftrightarrow$E.

$\leftrightarrow$E

a.	$\mathcal{P} \leftrightarrow \mathcal{Q}$		a.	$\mathcal{P} \leftrightarrow \mathcal{Q}$	
b.	$\mathcal{P}$		b.	$\mathcal{Q}$	
	$\mathcal{Q}$	a,b $\leftrightarrow$E		$\mathcal{P}$	a,b $\leftrightarrow$E

$\mathcal{P} \leftrightarrow \mathcal{Q}$ thus works like either $\mathcal{P} \rightarrow \mathcal{Q}$ or $\mathcal{Q} \rightarrow \mathcal{P}$. Intuitively given $\mathcal{P}$ if and *only if* $\mathcal{Q}$, then if $\mathcal{P}$ is true, $\mathcal{Q}$ is true. And given $\mathcal{P}$ *if* and only if $\mathcal{Q}$, then if $\mathcal{Q}$ is true $\mathcal{P}$ is true. On tables, if $\mathcal{P} \leftrightarrow \mathcal{Q}$ is true, then $\mathcal{P}$ and $\mathcal{Q}$ have the same truth value. So if $\mathcal{P} \leftrightarrow \mathcal{Q}$ is true and $\mathcal{P}$ is true, $\mathcal{Q}$ is true as well; and if $\mathcal{P} \leftrightarrow \mathcal{Q}$ is true and $\mathcal{Q}$ is true, $\mathcal{P}$ is true as well.

Given that $\mathcal{P} \leftrightarrow \mathcal{Q}$ can be exploited like $\mathcal{P} \rightarrow \mathcal{Q}$ or $\mathcal{Q} \rightarrow \mathcal{P}$, it is not surprising that introducing $\mathcal{P} \leftrightarrow \mathcal{Q}$ is like introducing both $\mathcal{P} \rightarrow \mathcal{Q}$ and $\mathcal{Q} \rightarrow \mathcal{P}$. The input to $\leftrightarrow$I is *two* subderivations. Given an accessible subderivation beginning with assumption $\mathcal{P}$ on line *a* and ending with conclusion $\mathcal{Q}$ against its scope line on *b*, and an accessible subderivation beginning with assumption $\mathcal{Q}$ on line *c* and ending with conclusion $\mathcal{P}$ against its scope line on *d*, one may conclude $\mathcal{P} \leftrightarrow \mathcal{Q}$ with justification, *a-b,c-d* $\leftrightarrow$I.

$\leftrightarrow$I

a.	| $\mathcal{P}$	A (g, $\leftrightarrow$I)
b.	| $\mathcal{Q}$	
c.	| $\mathcal{Q}$	A (g, $\leftrightarrow$I)
d.	| $\mathcal{P}$	
	$\mathcal{P} \leftrightarrow \mathcal{Q}$	a-b,c-d $\leftrightarrow$I

Intuitively, if an assumption $\mathcal{P}$ leads to $\mathcal{Q}$ and the assumption $\mathcal{Q}$ leads to $\mathcal{P}$, then we know that *if* $\mathcal{P}$ then $\mathcal{Q}$, and *if* $\mathcal{Q}$ then $\mathcal{P}$ — which is to say that $\mathcal{P}$ if and only if $\mathcal{Q}$. On truth tables, if there is a sententially valid argument from some other premises together with assumption $\mathcal{P}$, to conclusion $\mathcal{Q}$, then there is no row where those other premises are true and assumption $\mathcal{P}$ is true and $\mathcal{Q}$ is false; and if there is a sententially valid argument from those other premises together with assumption $\mathcal{Q}$ to conclusion $\mathcal{P}$, then there is no row where those other premises are true and the assumption $\mathcal{Q}$ is true and $\mathcal{P}$ is false; so on rows where the other premises are true, $\mathcal{P}$ and $\mathcal{Q}$ do not have different values, and the biconditional $\mathcal{P} \leftrightarrow \mathcal{Q}$ is true.

Here are a couple of examples. The first is straightforward, and exercises both the $\leftrightarrow$I and $\leftrightarrow$E rules. We show, $A \leftrightarrow B, B \leftrightarrow C \vdash_{ND} A \leftrightarrow C$.

(X)	1.	$A \leftrightarrow B$	P
	2.	$B \leftrightarrow C$	P
	3.	| A	A (g, $\leftrightarrow$I)
	4.	| B	1,3 $\leftrightarrow$E
	5.	| C	2,4 $\leftrightarrow$E
	6.	| C	A (g, $\leftrightarrow$I)
	7.	| B	2,6 $\leftrightarrow$E
	8.	| A	1,7 $\leftrightarrow$E
	9.	$A \leftrightarrow C$	3-5,6-8 $\leftrightarrow$I

Our original goal is $A \leftrightarrow C$. So it is natural to set up subderivations to get it by $\leftrightarrow$I. Once we have done this, the subderivations are easily completed by applications of $\leftrightarrow$E.

Here is an interesting case that again exercises both rules. We show, $A \leftrightarrow (B \leftrightarrow C), C \vdash_{ND} A \leftrightarrow B$.

ND Quick Reference (Sentential)

R (*reiteration*)

a.	$\mathcal{P}$	
	$\mathcal{P}$	a R

$\sim$I (*negation intro*)

a.	$\quad\mathcal{P}$	A (c, $\sim$I)
b.	$\quad\mathcal{Q} \wedge \sim\mathcal{Q}$ ($\bot$)	
	$\sim\mathcal{P}$	a-b $\sim$I

$\sim$E (*negation exploit*)

a.	$\quad\sim\mathcal{P}$	A (c, $\sim$E)
b.	$\quad\mathcal{Q} \wedge \sim\mathcal{Q}$ ($\bot$)	
	$\mathcal{P}$	a-b $\sim$E

$\wedge$I (*conjunction intro*)

a.	$\mathcal{P}$	
b.	$\mathcal{Q}$	
	$\mathcal{P} \wedge \mathcal{Q}$	a,b $\wedge$I

$\wedge$E (*conjunction exploit*)

a.	$\mathcal{P} \wedge \mathcal{Q}$	
	$\mathcal{P}$	a $\wedge$E

$\wedge$E (*conjunction exploit*)

a.	$\mathcal{P} \wedge \mathcal{Q}$	
	$\mathcal{Q}$	a $\wedge$E

$\vee$I (*disjunction intro*)

a.	$\mathcal{P}$	
	$\mathcal{P} \vee \mathcal{Q}$	a $\vee$I

$\vee$I (*disjunction intro*)

a.	$\mathcal{P}$	
	$\mathcal{Q} \vee \mathcal{P}$	a $\vee$I

$\vee$E (*disjunction exploit*)

a.	$\mathcal{P} \vee \mathcal{Q}$	
b.	$\quad\mathcal{P}$	A (g, a$\vee$E)
c.	$\quad\mathcal{C}$	
d.	$\quad\mathcal{Q}$	A (g, a$\vee$E)
e.	$\quad\mathcal{C}$	
	$\mathcal{C}$	a,b-c,d-e $\vee$E

$\rightarrow$I (*conditional intro*)

a.	$\quad\mathcal{P}$	A (g, $\rightarrow$I)
b.	$\quad\mathcal{Q}$	
	$\mathcal{P} \rightarrow \mathcal{Q}$	a-b $\rightarrow$I

$\rightarrow$E (*conditional exploit*)

a.	$\mathcal{P} \rightarrow \mathcal{Q}$	
b.	$\mathcal{P}$	
	$\mathcal{Q}$	a,b $\rightarrow$E

$\leftrightarrow$I (*biconditional intro*)

a.	$\quad\mathcal{P}$	A (g, $\leftrightarrow$I)
b.	$\quad\mathcal{Q}$	
c.	$\quad\mathcal{Q}$	A (g, $\leftrightarrow$I)
d.	$\quad\mathcal{P}$	
	$\mathcal{P} \leftrightarrow \mathcal{Q}$	a-b,c-d $\leftrightarrow$I

$\leftrightarrow$E (*biconditional exploit*)

a.	$\mathcal{P} \leftrightarrow \mathcal{Q}$	
b.	$\mathcal{P}$	
	$\mathcal{Q}$	a,b $\leftrightarrow$E

$\leftrightarrow$E (*biconditional exploit*)

a.	$\mathcal{P} \leftrightarrow \mathcal{Q}$	
b.	$\mathcal{Q}$	
	$\mathcal{P}$	a,b $\leftrightarrow$E

derived rule: $\bot$I (*bottom intro*)

a.	$\mathcal{Q}$	
b.	$\sim\mathcal{Q}$	
	$\bot$	a,b $\bot$I

(Y)

1.	$A \leftrightarrow (B \leftrightarrow C)$	P
2.	C	P
3.	$\quad A$	A (g, $\leftrightarrow$I)
4.	$\quad B \leftrightarrow C$	1,3 $\leftrightarrow$E
5.	$\quad B$	4,2 $\leftrightarrow$E
6.	$\quad B$	A (g, $\leftrightarrow$I)
7.	$\quad\quad B$	A (g, $\leftrightarrow$I)
8.	$\quad\quad C$	2 R
9.	$\quad\quad C$	A (g, $\leftrightarrow$I)
10.	$\quad\quad B$	6 R
11.	$\quad B \leftrightarrow C$	7-8,9-10 $\leftrightarrow$I
12.	$\quad A$	1,11 $\leftrightarrow$E
13.	$A \leftrightarrow B$	3-5,6-12 $\leftrightarrow$I

We begin by setting up the subderivations to get $A \leftrightarrow B$ by $\leftrightarrow$I. This first is easily completed with a couple applications of $\leftrightarrow$E. To reach the goal for the second by means of the premise (1) we need $B \leftrightarrow C$ as our second "card." So we set up to reach *that*. As it happens, the extra subderivations at (7) - (8) and (9) - (10) are easy to complete. Again, if you have followed so far, you are doing well. We will be in a better position to *create* such derivations after our discussion of strategy.

So much for the rules for this sentential part of *ND*. Before we turn in the next sections to strategy, let us note a couple of features of the rules that may so-far have gone without notice. First, premises are not always necessary for *ND* derivations. Thus, for example, $\vdash_{ND} A \rightarrow A$.

(Z)

1.	$\quad A$	A (g, $\rightarrow$I)
2.	$\quad A$	1 R
3.	$A \rightarrow A$	1-2 $\rightarrow$I

If there are no premises, do not panic! Begin in the usual way. In this case, the original goal is $A \rightarrow A$. So we set up to obtain it by $\rightarrow$I. And the subderivation is particularly simple. Notice that our derivation of $A \rightarrow A$ corresponds to the fact from truth tables that $\vDash_s A \rightarrow A$. And we *need* to be able to derive $A \rightarrow A$ from no premises if there is to be the right sort of correspondence between derivations in *ND* and semantic validity — if we are to have $\Gamma \vDash \mathcal{P}$ iff $\Gamma \vdash_{ND} P$.

Second, observe again that every subderivation comes with an exit strategy. The exit strategy says whether you intend to complete the subderivation with a particular goal, or by obtaining a contradiction, and then how the subderivation is to be used once complete. There are just five rules which appeal to a subderivation: →I, ∼I, ∼E, ∨E, and ↔I. You will complete the subderivation, and then use it by one of these rules. So these are the *only* rules which may appear in an exit strategy. If you do not understand this, then you need to go back and think about the rules until you do.

Finally, it is worth noting a strange sort of case, with application to rules that can take more than one input of the same type. Consider a simple demonstration that $A \vdash_{ND} A \wedge A$. We might proceed as in (AA) on the left,

(AA)

1.	A	P
2.	A	1 R
3.	$A \wedge A$	1,2 ∧I

(AB)

1.	A	P
3.	$A \wedge A$	1,1 ∧I

We begin with A, reiterate so that A appears on different lines, and apply ∧I. But we might have proceeded as in (AB) on the right. The rule requires an accessible line on which the left conjunct appears — which we have at (1), and an accessible line on which the right conjunct appears *which we also have* on (1). So the rule takes an input for the left conjunct and an input for the right — they just happen to be the same thing. A similar point applies to rules ∨E and ↔I which take more than one subderivation as input. Suppose we want to show $A \vee A \vdash_{ND} A$.[4]

(AC)

1.	$A \vee A$	P
2.	│ A	A (g, 1∨E)
3.	│ A	2 R
4.	│ A	A (g, 1∨E)
5.	│ A	4 R
6.	A	1,2-3,4-5 ∨E

(AD)

1.	$A \vee A$	P
2.	│ A	A (g, 1∨E)
3.	│ A	2 R
4.	A	1,2-3,2-3 ∨E

In (AC), we begin in the usual way to get the main goal by ∨E. This leads to the subderivations (2) - (3) and (4) - (5), the first moving from the left disjunct to the goal, and the second from the right disjunct to the goal. But the left and right disjuncts are the same! So we might have simplified as in (AD). ∨E still requires three inputs: First an accessible disjunction, which we find on (1); second an accessible subderivation which moves from the left disjunct to the goal, which we find on (2) - (3); third a subderivation which moves from the right disjunct to the goal — *but we have this on* (2) - (3). So the justification at (4) of (AD) appeals to the three relevant facts, by appeal to the same subderivation twice. Similarly one could imagine a quick-and-dirty demonstration that $\vdash_{ND} A \leftrightarrow A$.

[4] I am reminded of an irritating character in *Groundhog Day* who repeatedly asks, "Am I right or am I right?" If he implies that the disjunction is true, it follows that he is right.

E6.11. Complete the following derivations by filling in justifications for each line.

a.
1. $| \underline{A \leftrightarrow B}$
2. $| \; | \underline{A}$
3. $| \; | B$
4. $| A \rightarrow B$

b.
1. $| A \leftrightarrow B$
2. $| \underline{\sim B}$
3. $| \; | \underline{A}$
4. $| \; | B$
5. $| \; | \bot$
6. $| \sim A$

c.
1. $| \; | \underline{A \leftrightarrow \sim A}$
2. $| \; | \; | \underline{A}$
3. $| \; | \; | \sim A$
4. $| \; | \; | \bot$
5. $| \; | \sim A$
6. $| \; | A$
7. $| \; | \bot$
8. $| \sim (A \leftrightarrow \sim A)$

d.
1. $| \; | \underline{A}$
2. $| \; | \; | \underline{\sim A}$
3. $| \; | \; | A$
4. $| \; | \sim A \rightarrow A$
5. $| \; | \underline{\sim A \rightarrow A}$
6. $| \; | \; | \underline{\sim A}$
7. $| \; | \; | A$
8. $| \; | \; | \bot$
9. $| \; | A$
10. $| A \leftrightarrow (\sim A \rightarrow A)$

e.

1.	$\sim A$
2.	$\sim B$
3.	| A
4.	| | $\sim B$
5.	| | $\bot$
6.	| B
7.	| B
8.	| | $\sim A$
9.	| | $\bot$
10.	| A
11.	$A \leftrightarrow B$

E6.12. Each of the following are not legitimate *ND* derivations. In each case, explain why.

a.

1.	A	P
2.	B	P
3.	$A \leftrightarrow B$	1,2 $\leftrightarrow$I

b.

1.	$A \rightarrow B$	P
2.	B	P
3.	A	1,2 $\rightarrow$E

*c.

1.	$A \leftrightarrow B$	P
2.	A	1 $\leftrightarrow$E

d.

1.	B	P
2.	| A	A (g, $\leftrightarrow$I)
3.	| B	1 R
4.	| B	A (g, $\leftrightarrow$I)
5.	| A	2 R
6.	$A \leftrightarrow B$	2-3,4-5 $\leftrightarrow$I

e.

1.	$\sim A$	P
2.	$\quad B$	A (g, $\rightarrow$I)
3.	$\quad\quad \sim A$	A (g, $\leftrightarrow$I)
4.	$\quad\quad B$	2 R
5.	$\quad B$	2 R
6.	$B \rightarrow B$	2-5 $\rightarrow$I
7.	$\quad B$	A (g, $\leftrightarrow$I)
8.	$\quad \sim A$	1 R
9.	$\sim A \leftrightarrow B$	3-4,7-8 $\leftrightarrow$I

E6.13. Produce derivations to show each of the following.

*a. $(A \wedge B) \leftrightarrow A \vdash_{ND} A \rightarrow B$

b. $A \leftrightarrow (A \vee B) \vdash_{ND} B \rightarrow A$

c. $A \leftrightarrow B, B \leftrightarrow C, C \leftrightarrow D, \sim A \vdash_{ND} \sim D$

d. $A \leftrightarrow B \vdash_{ND} (A \rightarrow B) \wedge (B \rightarrow A)$

*e. $A \leftrightarrow (B \wedge C), B \vdash_{ND} A \leftrightarrow C$

f. $(A \rightarrow B) \wedge (B \rightarrow A) \vdash_{ND} (A \leftrightarrow B)$

g. $A \rightarrow (B \leftrightarrow C) \vdash_{ND} (A \wedge B) \leftrightarrow (A \wedge C)$

h. $A \leftrightarrow B, C \leftrightarrow D \vdash_{ND} (A \wedge C) \leftrightarrow (B \wedge D)$

i. $\vdash_{ND} A \leftrightarrow A$

j. $\vdash_{ND} (A \wedge B) \leftrightarrow (B \wedge A)$

*k. $\vdash_{ND} \sim\sim A \leftrightarrow A$

l. $\vdash_{ND} (A \leftrightarrow B) \rightarrow (B \leftrightarrow A)$

m. $(A \wedge B) \leftrightarrow (A \wedge C) \vdash_{ND} A \rightarrow (B \leftrightarrow C)$

n. $\sim A \rightarrow B, A \rightarrow \sim B \vdash_{ND} \sim A \leftrightarrow B$

o. $A, B \vdash_{ND} \sim A \leftrightarrow \sim B$

6.2.4 Strategies for a Goal

It is natural to introduce derivation rules, as we have, with relatively simple cases. And you may or may not have been able to see from the start in some cases how derivations would go. But derivations are not always so simple, and (short of genius) nobody can always see how they go. Perhaps this has already been an issue! So we want to think about derivation strategies. As we shall see later, for the quantificational case at least, it is not *possible* to produce a mechanical algorithm adequate to complete every completable derivation. However, as with chess or other games of strategy, it is possible to say a good deal about how to approach problems effectively. We have said quite a bit already. In this section, we pull together some of the themes, and present the material more systematically.

For natural derivation systems, the overriding strategy is to *work goal directedly*. What you do at any stage is directed primarily, not by what you have, but by where you want to be. Suppose you are trying to show that $\Gamma \vdash_{ND} \mathcal{P}$. You are given $\mathcal{P}$ as your goal. Perhaps it is tempting to begin by using E-rules to "see what you can get" from the members of Γ. There is nothing wrong with a bit of this in order to simplify your premises (like arranging the cards in your hand into some manageable order), but the main work of doing a derivation does not begin until you focus on the goal. This is not to say that your premises play no role in strategic thinking. Rather, it is to rule out doing things with them which are not purposefully directed at the end. In the ordinary case, applying the strategies for your goal dictates some new goal; applying strategies for this new goal dictates another; and so forth, until you come to a goal that is easily achieved.

The following *strategies for a goal* are arranged in rough priority order:

SG
1. If accessible lines contain explicit contradiction, use $\sim$E to reach goal.
2. Given an accessible formula with main operator $\vee$, use $\vee$E to reach goal.
3. If goal is "in" accessible lines (set goals and) attempt to exploit it out.
4. To reach goal with main operator $\star$, use $\star$I (careful with $\vee$).
5. Try $\sim$E (especially for atomics and sentences with $\vee$ as main operator).

If a high priority strategy applies, use it. If one does not apply, simply "fall through" to the next. The priority order is not necessarily a frequency order. The frequency will likely be something like SG4, SG3, SG5, SG2, SG1. But high priority strategies are such that you should adopt them if they are available — even though most often you will fall through to ones that are more frequently used. I take up the strategies in the priority order.

SG1 *If accessible lines contain explicit contradiction, use* ∼*E to reach goal*. For goal $\mathcal{B}$, with an explicit contradiction accessible, you can simply *assume* $\sim\mathcal{B}$, use your contradiction, and conclude $\mathcal{B}$.

given

a.	$\mathcal{A}$	
b.	$\sim\mathcal{A}$	
	$\mathcal{B}$	(goal)

use

a.	$\mathcal{A}$	
b.	$\sim\mathcal{A}$	
c.	$\quad\sim\mathcal{B}$	A (c, ∼E)
d.	$\quad\bot$	a,b ⊥I
	$\mathcal{B}$	c-d ∼E

That is it! No matter what your goal is, given an accessible contradiction, you can reach that goal by ∼E. Since this strategy always delivers, you should jump on it whenever it is available. As an example, try to show, $A, \sim A \vdash_{ND} (R \wedge S) \rightarrow T$. Your derivation need not involve →I. Hint: I mean it! This section will be far more valuable if you work these examples, and so think through the steps. Here it is in two stages.

(AE)

1.	A	P
2.	$\sim A$	P
3.	$\quad\sim[(R \vee S) \rightarrow T]$	A (c, ∼E)
	$(R \vee S) \rightarrow T$	

1.	A	P
2.	$\sim A$	P
3.	$\quad\sim[(R \vee S) \rightarrow T]$	A (c, ∼E)
4.	$\quad\bot$	1,2 ⊥I
5.	$(R \vee S) \rightarrow T$	3-4 ∼E

As soon as we see the accessible contradiction, we assume the negation of our goal, with a plan to exit by ∼E. This is accomplished on the left. Then it is a simple matter of applying the contradiction, and going to the conclusion by ∼E.

For this strategy, it is not required that accessible lines "contain" a contradiction only when you already have $\mathcal{Q}$ and $\sim\mathcal{Q}$ for ⊥I. However, the intent is that it should be no real work to obtain them. Perhaps an application of ∧E or the like does the job. It should be possible to obtain the contradiction directly by some E-rule(s). If you can do this, then your derivation is over: assuming the opposite, applying the rules, and then ∼E reaches the goal. If there is no simple way to obtain a contradiction, fall through to the next strategy.

SG2 *Given an accessible formula with main operator* ∨*, use* ∨*E to reach goal*. As suggested above, you may prefer to avoid ∨E. But this is a mistake — ∨E is your friend! Suppose you have some accessible lines including a disjunction $\mathcal{A} \vee \mathcal{B}$ with goal $\mathcal{C}$. If you go for *that very goal* by ∨E, the result is a pair of subderivations with goal $\mathcal{C}$ — where, in the one case, all those very same accessible lines *and* $\mathcal{A}$ are accessible, and in the other case, all those very same lines *and* $\mathcal{B}$ are accessible. So, in each subderivation, you can only be better off in your attempt to reach $\mathcal{C}$.

given

a.	$\mathcal{A} \lor \mathcal{B}$	
	$\mathcal{C}$	(goal)

use

a.	$\mathcal{A} \lor \mathcal{B}$	
b.	$\mathcal{A}$	A (g, a$\lor$E)
c.	$\mathcal{C}$	(goal)
d.	$\mathcal{B}$	A (g, a$\lor$E)
e.	$\mathcal{C}$	(goal)
	$\mathcal{C}$	a,b-c,d-e $\lor$E

As an example, try to show, $A \to B,\ A \lor (A \land B) \vdash_{ND} A \land B$. Try showing it without $\lor$E! Here is the derivation in stages.

(AF)

1.	$A \to B$	P
2.	$A \lor (A \land B)$	P
3.	A	A (g, 2$\lor$E)
	$A \land B$	
	$A \land B$	A (g, 2$\lor$E)
	$A \land B$	
	$A \land B$	

1.	$A \to B$	P
2.	$A \lor (A \land B)$	P
3.	A	A (g, 2$\lor$E)
4.	B	1,3 $\to$E
5.	$A \land B$	3,4 $\land$I
6.	$A \land B$	A (g, 2$\lor$E)
7.	$A \land B$	6 R
8.	$A \land B$	1,2-5,6-7 $\lor$E

When we start, there is no accessible contradiction. So we fall through to SG2. Since a premise has main operator $\lor$, we set up to get the goal by $\lor$E. This leads to a pair of simple subderivations. Once we do this, we treat the disjunction as effectively "used up" so that SG2 does not apply to it again. Notice that there is almost nothing one *could* do except set up this way — and that once you do, it is easy!

SG3 *If goal is "in" accessible lines (set goals and) attempt to exploit it out*. In most derivations, you will work toward goals which are successively closer to what can be obtained directly from accessible lines. And you finally come to a goal which can be obtained directly. If it can be obtained directly, do so! In some cases, however, you will come to a stage where your goal exists in accessible lines, but can be obtained only by means of some other result. In this case, you can set that other result as a *new* goal. A typical case is as follows.

given

a.	$\mathcal{A} \to \mathcal{B}$	
	$\mathcal{B}$	(goal)

use

a.	$\mathcal{A} \to \mathcal{B}$	
b.	$\mathcal{A}$	(goal)
	$\mathcal{B}$	a,b $\to$E

The $\mathcal{B}$ exists in the premises. You cannot get it without the $\mathcal{A}$. So you set $\mathcal{A}$ as a new goal and use it to get the $\mathcal{B}$. It is impossible to represent all the cases where this strategy applies. The idea is that the complete goal exists in accessible lines, and can either be obtained directly by an E-rule, or by an E-rule with some new goal. Observe that the strategy would not apply in case you have $A \to B$ and are going for A. Then the goal exists as part of a premise all right. But there is no obvious result such that obtaining it would give you a way to exploit $A \to B$ to get the A.

As an example, let us try to show $(A \to B) \wedge (B \to C)$, $(L \leftrightarrow S) \to A$, $(L \leftrightarrow S) \wedge H \vdash_{ND} C$. Here is the derivation in four stages.

(AG)

1.	$(A \to B) \wedge (B \to C)$	P
2.	$(L \leftrightarrow S) \to A$	P
3.	$(L \leftrightarrow S) \wedge H$	P
4.	$B \to C$	1 ∧E
	B	
	C	4,_ →E

1.	$(A \to B) \wedge (B \to C)$	P
2.	$(L \leftrightarrow S) \to A$	P
3.	$(L \leftrightarrow S) \wedge H$	P
4.	$B \to C$	1 ∧E
5.	$A \to B$	1 ∧E
	A	
	B	5,_ →E
	C	4,_ →E

The original goal C exists in the premises, as the consequent of the right conjunct of (1). It is easy to isolate the $B \to C$, but this leaves us with the B as a new goal to get the C. B also exists in the premises, as the consequent of the left conjunct of (1). Again, it is easy to isolate $A \to B$, but this leaves us with A as a new goal. We are not in a position to fill in the entire justification for our new goals, but there is no harm filling in what we can, to remind us where we are going. So far, so good.

1.	$(A \to B) \wedge (B \to C)$	P
2.	$(L \leftrightarrow S) \to A$	P
3.	$(L \leftrightarrow S) \wedge H$	P
4.	$B \to C$	1 ∧E
5.	$A \to B$	1 ∧E
	$L \leftrightarrow S$	
	A	2,_ →E
	B	5,_ →E
	C	4,_ →E

1.	$(A \to B) \wedge (B \to C)$	P
2.	$(L \leftrightarrow S) \to A$	P
3.	$(L \leftrightarrow S) \wedge H$	P
4.	$B \to C$	1 ∧E
5.	$A \to B$	1 ∧E
6.	$L \leftrightarrow S$	3 ∧E
7.	A	2,6 →E
8.	B	5,7 →E
9.	C	4,8 →E

But A also exists in the premises, as the consequent of (2); to get it, we set $L \leftrightarrow S$ as a goal. But $L \leftrightarrow S$ exists in the premises, and is easy to get by ∧E. So we complete the derivation with the steps that motivated the subgoals in the first place. Observe the way we move from one goal to the next, until finally there is a stage where SG3 applies in its simplest form, so that $L \leftrightarrow S$ is obtained directly.

SG4 *To reach goal with main operator* $\star$*, use* $\star$*I (careful with* $\vee$*).* This is the most frequently used strategy, the one most likely to structure your derivation as a whole. $\sim$E to the side, the basic structure of I-rules and E-rules in *ND* gives you just one way to generate a formula with main operator $\star$, whatever that may be. In the ordinary case, then, you can *expect* to obtain a formula with main operator $\star$ by the corresponding I-rule. Thus, for a typical example,

given

	$\mathcal{A} \to \mathcal{B}$	(goal)

use

a.	$\mathcal{A}$	A (g, $\to$I)
b.	$\mathcal{B}$	(goal)
	$\mathcal{A} \to \mathcal{B}$	a-b $\to$I

Again, it is difficult to represent all the cases where this strategy might apply. It makes sense to consider it for formulas with any main operator. Be cautious, however, for formulas with main operator $\vee$. There are cases where it is possible to prove a disjunction, but not to prove it by $\vee$I — as one might have conclusive reason to believe the butler *or* the maid did it, without conclusive reason to believe the butler did it, or conclusive reason to believe the maid did it (perhaps the butler and maid were the only ones with means and motive). You should consider the strategy for $\vee$. But it does not always work.

As an example, let us show $D \vdash_{ND} A \to (B \to (C \to D))$. Here is the derivation in four stages.

(AH)

1.	D	P
2.	A	A (g, $\to$I)
	$B \to (C \to D)$	
	$A \to (B \to (C \to D))$	2-_ $\to$I

1.	D	P
2.	A	A (g, $\to$I)
3.	B	A (g, $\to$I)
	$C \to D$	
	$B \to (C \to D)$	3-_ $\to$I
	$A \to (B \to (C \to D))$	2-_ $\to$I

Initially, there is no contradiction or disjunction in the premises, and neither do we see the goal. So we fall through to strategy SG4 and, since the main operator of the goal is $\to$, set up to get it by $\to$I. This gives us $B \to (C \to D)$ as a new goal. Since this has main operator $\to$, and it remains that other strategies do not apply, we fall through to SG4, and set up to get it by $\to$I. This gives us $C \to D$ as a new goal.

1.	D	P
2.	$\quad A$	A (g, →I)
3.	$\quad\quad B$	A (g, →I)
4.	$\quad\quad\quad C$	A $g, \to I)$
	$\quad\quad\quad D$	
	$\quad\quad C \to D$	4-__ →I
	$\quad B \to (C \to D)$	3-__ →I
	$A \to (B \to (C \to D))$	2-__ →I

1.	D	P
2.	$\quad A$	A (g, →I)
3.	$\quad\quad B$	A (g, →I)
4.	$\quad\quad\quad C$	A $g, \to I)$
5.	$\quad\quad\quad D$	1 R
6.	$\quad\quad C \to D$	4-5 →I
7.	$\quad B \to (C \to D)$	3-6 →I
8.	$A \to (B \to (C \to D))$	2-7 →I

As before, with $C \to D$ as the goal, there is no contradiction on accessible lines, no accessible formula has main operator ∨, and the goal does not itself appear on accessible lines. Since the main operator is →, we set up again to get it by →I. This gives us D as a new subgoal. But D does exist on an accessible line. Thus we are faced with a particularly simple instance of strategy SG3. To complete the derivation, we simply reiterate D from (1), and follow our exit strategies as planned.

SG5 *Try* ∼E (especially for atomics and sentences with ∨ as main operator). The previous strategy has no application to atomics, because they *have* no main operator, and we have suggested that it is problematic for disjunctions. This last strategy applies particularly in those cases. So it is applicable in cases where other strategies seem not to apply.

given

$\mathscr{A}$	(goal)

use

a.	$\quad \sim\mathscr{A}$	A (c, ∼E)
b.	$\quad \bot$	
	$\mathscr{A}$	a-b ∼E

It is possible to obtain *any* formula by ∼E, by assuming the negation of it and going for a contradiction. So this strategy is generally applicable. And it cannot hurt: If you could have reached the goal anyway, you can obtain the goal $\mathscr{A}$ under the assumption, and then use *it* for a contradiction with the assumed $\sim\mathscr{A}$ — which lets you exit the assumption with the $\mathscr{A}$ you would have had anyway. And the assumption may help: for, as with ∨E, in going for the contradiction you have whatever accessible lines you had before, *plus* the new assumption. And, in many cases, the assumption puts you in a position to make progress you would not have been able to make before.

As a simple example of the strategy, try showing, $\sim A \to B, \sim B \vdash_{ND} A$. Here is the derivation in two stages.

(AI)

1.	$\sim A \rightarrow B$	P
2.	$\sim B$	P
3.	$\quad \sim A$	A (c, $\sim$E)
	$\quad \bot$	
	A	3- _ $\sim$E

1.	$\sim A \rightarrow B$	P
2.	$\sim B$	P
3.	$\quad \sim A$	A (c, $\sim$E)
4.	$\quad B$	1,3 $\rightarrow$E
5.	$\quad \bot$	4,2 $\bot$I
6.	A	3-5 $\sim$E

Sometimes the occasion between this strategy and SG1 can seem obscure (and, in the end, it may not be all that important to separate them). However, for the first strategy, accessible lines *by themselves* are sufficient for a contradiction. In this example, from the premises we have $\sim B$, but cannot get the B and so do not have a contradiction from the premises alone. So SG1 does not apply. There is no formula with main operator $\vee$. Similarly, though $\sim A$ is in the antecedent of (1), there is no obvious way to exploit the premise to isolate the A; so we do not see the goal in the relevant form in the premises. The goal A has no operators, so it has no main operator and strategy SG4 does not apply. So we fall through to strategy SG5, and set up to get the goal by $\sim$E. In this case, the subderivation is particularly easy to complete. Perhaps the case is too easy. Still, in contrast to SG1, the contradiction does not become available until after you make the assumption. In the case of SG1, it is the prior availability of the contradiction that drives your assumption.

Here is an extended example which combines a number of the strategies considered so far. We show that $B \vee A \vdash_{ND} \sim A \rightarrow B$. You want especially to absorb the *mode of thinking* about this case as a way to approach exercises.

(AJ)

1.	$B \vee A$	P
	$\sim A \rightarrow B$	

There is no contradiction in accessible premises; so strategy SG1 is inapplicable. Strategy SG2 tells us to go for the goal by $\vee$E. Another option is to fall through to SG4 and go for $\sim A \rightarrow B$ by $\rightarrow$I and then apply $\vee$E to get the B, but $\rightarrow$I has lower priority, and let us follow the official procedure.

1.	$B \vee A$	P	
2.	$\mid B$	A (g, $1\vee$E)	
	$\mid \sim A \rightarrow B$		Given an accessible line with main operator $\vee$, use $\vee$E to reach goal.
	$\mid A$	A (g, $1\vee$E)	
	$\mid \sim A \rightarrow B$		
	$\sim A \rightarrow B$	$1, _, _ \; \vee$E	

Having set up for $\vee$E on line (1), we treat $B \vee A$ as effectively "used up" and so out of the picture. Concentrating, for the moment, on the first subderivation, there is no contradiction on accessible lines; neither is there another accessible disjunction; and the goal is not in the premises. So we fall through to SG4.

1.	$B \vee A$	P	
2.	$\mid B$	A (g, $1\vee$E)	
3.	$\mid \mid \sim A$	A (g, $\rightarrow$I)	
	$\mid \mid B$		
	$\mid \sim A \rightarrow B$	$3\text{-}_ \rightarrow$I	To reach goal with main operator $\rightarrow$, use $\rightarrow$I.
	$\mid A$	A (g, $1\vee$E)	
	$\mid \sim A \rightarrow B$		
	$\sim A \rightarrow B$	$1, _, _ \; \vee$E	

In this case, the subderivation is easy to complete. The new goal, B exists as such in the premises. So we are faced with a simple instance of SG3, and so can complete the subderivation.

1.	$B \vee A$	P	
2.	$\mid B$	A (g, $1\vee$E)	
3.	$\mid \mid \sim A$	A (g, $\rightarrow$I)	
4.	$\mid \mid B$	2 R	The first subderivation is completed by reiterating B from line (2), and following the exit strategy.
5.	$\mid \sim A \rightarrow B$	3-4 $\rightarrow$I	
6.	$\mid A$	A (g, $1\vee$E)	
	$\mid \sim A \rightarrow B$		
	$\sim A \rightarrow B$	$1, _, _ \; \vee$E	

For the second main subderivation tick off in your head: there is no accessible contradiction; neither is there another accessible formula with main operator ∨; and the goal is not in the premises. So we fall through to strategy SG4.

```
 1. | B ∨ A            P
    |---
 2. | | B              A (g, 1∨E)
    | |---
 3. | | | ∼A           A (g, →I)
    | | |---
 4. | | | B            2 R
 5. | | ∼A → B         3-4 →I
    |
 6. | | A              A (g, 1∨E)         To reach goal with main operator →, use →I.
    | |---
 7. | | | ∼A           A (g, →I)
    | | |---
    | | |
    | | | B
    | | ∼A → B         7-_ →I
    | ∼A → B           1, _, _ ∨E
```

In this case, there *is* an accessible contradiction at (6) and (7). So SG1 applies, and we are in a position to complete the derivation as follows.

```
 1. | B ∨ A            P
    |---
 2. | | B              A (g, 1∨E)
    | |---
 3. | | | ∼A           A (g, →I)
    | | |---
 4. | | | B            2 R
 5. | | ∼A → B         3-4 →I
    |
 6. | | A              A (g, 1∨E)         If accessible lines contain explicit contradiction,
    | |---                                use ∼E to reach goal.
 7. | | | ∼A           A (g, →I)
    | | |---
 8. | | | | ∼B         A (c, ∼E)
    | | | |---
 9. | | | | ⊥          6,7 ⊥I
10. | | | B            8-9 ∼E
11. | | ∼A → B         7-10 →I
12. | ∼A → B           1,2-5,6-11 ∨E
```

This derivation is fairly complicated! But we did not need to see how the whole thing would go from the start. Indeed, it is hard to see how one could do so. Rather it was enough to see, at each stage, what to do next. That is the beauty of our goal-oriented approach.

A couple of final remarks before we turn to exercises: First, as we have said from the start, assumptions are only introduced in conjunction with exit strategies. This almost requires goal-directed thinking. And it is important to see how pointless are assumptions without an exit strategy! Results inside subderivations cannot be used for a final conclusion except insofar as there is a way to exit the subderivation and use it whole. So the point of the strategy is to ensure that the subderivation has a use for getting where you want to go.

Second, in going for a contradiction, as with SG4 or SG5, the new goal is not a definite formula — any contradiction is sufficient for the rule and for a derivation of $\bot$. So the strategies for a goal do not directly apply. This motivates the "strategies for a contradiction" of the next section. For now, I will say just this: If there is a contradiction to be had, and you can reduce formulas on accessible lines to atomics and negated atomics, the contradiction *will* appear at that level. So one way to go for a contradiction is simply by applying E-rules to accessible lines, to generate what atomics and negated atomics you can.

Proof for the following theorems are left as exercises. You should not start them now, but wait for the assignment in E6.16. The first three may remind you of axioms from chapter 3 and the fourth has an application in **??**. The others foreshadow rules from the system *ND+*, which we will see shortly.

T6.1. $\vdash_{ND} \mathcal{P} \to (\mathcal{Q} \to \mathcal{P})$

T6.2. $\vdash_{ND} (\mathcal{O} \to (\mathcal{P} \to \mathcal{Q})) \to ((\mathcal{O} \to \mathcal{P}) \to (\mathcal{O} \to \mathcal{Q}))$

*T6.3. $\vdash_{ND} (\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})$

T6.4. $\mathcal{A} \to (\mathcal{B} \to \mathcal{C}), \mathcal{D} \to (\mathcal{C} \to \mathcal{E}), \mathcal{D} \to \mathcal{B} \vdash_{ND} \mathcal{A} \to (\mathcal{D} \to E)$

T6.5. $\mathcal{A} \to \mathcal{B}, \sim\mathcal{B} \vdash_{ND} \sim\mathcal{A}$

T6.6. $\mathcal{A} \to \mathcal{B}, \mathcal{B} \to \mathcal{C} \vdash_{ND} \mathcal{A} \to \mathcal{C}$

T6.7. $\mathcal{A} \vee \mathcal{B}, \sim\mathcal{A} \vdash_{ND} \mathcal{B}$

T6.8. $\mathcal{A} \vee \mathcal{B}, \sim\mathcal{B} \vdash_{ND} \mathcal{A}$

T6.9. $\mathcal{A} \leftrightarrow \mathcal{B}, \sim\mathcal{A} \vdash_{ND} \sim\mathcal{B}$

T6.10. $\mathcal{A} \leftrightarrow \mathcal{B}, \sim\mathcal{B} \vdash_{ND} \sim\mathcal{A}$

T6.11. $\vdash_{ND} (\mathcal{A} \wedge \mathcal{B}) \leftrightarrow (\mathcal{B} \wedge \mathcal{A})$

T6.12. $\vdash_{ND} (\mathcal{A} \leftrightarrow \mathcal{B}) \leftrightarrow (\mathcal{B} \leftrightarrow \mathcal{A})$

*T6.13. $\vdash_{ND} (\mathcal{A} \vee \mathcal{B}) \leftrightarrow (\mathcal{B} \vee \mathcal{A})$

T6.14. $\vdash_{ND} (\mathcal{A} \rightarrow \mathcal{B}) \leftrightarrow (\sim\mathcal{B} \rightarrow \sim\mathcal{A})$

T6.15. $\vdash_{ND} [\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})] \leftrightarrow [(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}]$

T6.16. $\vdash_{ND} [\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})] \leftrightarrow [(\mathcal{A} \wedge \mathcal{B}) \wedge \mathcal{C}]$

T6.17. $\vdash_{ND} [\mathcal{A} \vee (\mathcal{B} \vee \mathcal{C})] \leftrightarrow [(\mathcal{A} \vee \mathcal{B}) \vee \mathcal{C}]$

T6.18. $\vdash_{ND} \mathcal{A} \leftrightarrow \sim\sim\mathcal{A}$

T6.19. $\vdash_{ND} \mathcal{A} \leftrightarrow (\mathcal{A} \wedge \mathcal{A})$

T6.20. $\vdash_{ND} \mathcal{A} \leftrightarrow (\mathcal{A} \vee \mathcal{A})$

E6.14. For each of the following, (i) which goal strategy applies? and (ii) what is the next step? If the strategy calls for a new subgoal, show the subgoal; if it calls for a subderivation, set up the subderivation. In each case, *explain* your response. Hint: Each goal strategy applies once.

a.
1. $\sim A \vee B$ P
2. A P

B

b.
1. $J \wedge S$ P
2. $S \rightarrow K$ P

K

*c.
1. $\sim A \leftrightarrow B$ P

$B \leftrightarrow \sim A$

d.
1. $A \leftrightarrow \sim B$ P
2. $\sim A$ P

B

e.
1. $A \wedge B$ P
2. $\sim A$ P

$K \vee J$

E6.15. Produce derivations to show each of the following. No worked out answers are provided. However, if you get stuck, you will find strategy hints in the back.

*a. $A \leftrightarrow (A \rightarrow B) \vdash_{ND} A \rightarrow B$

*b. $(A \vee B) \rightarrow (B \leftrightarrow D), B \vdash_{ND} B \wedge D$

*c. $\sim(A \wedge C), \sim(A \wedge C) \leftrightarrow B \vdash_{ND} A \vee B$

*d. $A \wedge (C \wedge \sim B), (A \vee D) \rightarrow \sim E \vdash_{ND} \sim E$

*e. $A \rightarrow B, B \rightarrow C \vdash_{ND} A \rightarrow C$

*f. $(A \wedge B) \rightarrow (C \wedge D) \vdash_{ND} [(A \wedge B) \rightarrow C] \wedge [(A \wedge B) \rightarrow D]$

*g. $A \rightarrow (B \rightarrow C), (A \wedge D) \rightarrow E, C \rightarrow D \vdash_{ND} (A \wedge B) \rightarrow E$

*h. $(A \to B) \wedge (B \to C), [(D \vee E) \vee H] \to A, \sim(D \vee E) \wedge H \vdash_{ND} C$

*i. $A \to (B \wedge C), \sim C \vdash_{ND} \sim(A \wedge D)$

*j. $A \to (B \to C), D \to B \vdash_{ND} A \to (D \to C)$

*k. $A \to (B \to C) \vdash_{ND} \sim C \to \sim(A \wedge B)$

*l. $(A \wedge \sim B) \to \sim A \vdash_{ND} A \to B$

*m. $\sim B \leftrightarrow A, C \to B, A \wedge C \vdash_{ND} \sim K$

*n. $\sim A \vdash_{ND} A \to B$

*o. $\sim A \leftrightarrow \sim B \vdash_{ND} A \leftrightarrow B$

*p. $(A \vee B) \vee C, B \leftrightarrow C \vdash_{ND} C \vee A$

*q. $\vdash_{ND} A \to (A \vee B)$

*r. $\vdash_{ND} A \to (B \to A)$

*s. $\vdash_{ND} (A \leftrightarrow B) \to (A \to B)$

*t. $\vdash_{ND} (A \wedge \sim A) \to (B \wedge \sim B)$

*u. $\vdash_{ND} (A \to B) \to [(C \to A) \to (C \to B)]$

*v. $\vdash_{ND} [(A \to B) \wedge \sim B] \to \sim A$

*w. $\vdash_{ND} A \to [B \to (A \to B)]$

*x. $\vdash_{ND} \sim A \to [(B \wedge A) \to C]$

*y. $\vdash_{ND} (A \to B) \to [\sim B \to \sim(A \wedge D)]$

*E6.16. Produce derivations to demonstrate each of T6.1 - T6.20. This is a mix — some repetitious, some challenging! But, when we need the results later, we will be glad to have done them now. Hint: do not worry if one or two get a bit longer than you are used to — they should!

6.2.5 Strategies for a Contradiction

In going for a contradiction, the $\mathscr{Q}$ and $\sim\mathscr{Q}$ can be any sentence. So the strategies for reaching a definite goal do not apply. This motivates *strategies for a contradiction*. Again, the strategies are in rough priority order.

SC
1. Break accessible formulas down into atomics and negated atomics.
2. Given a disjunction in a subderivation for $\sim$E or $\sim$I, go for $\bot$ by $\vee$E.
3. Set as goal the opposite of some negation (something that cannot itself be broken down). Then apply strategies for a goal to reach it.
4. For some $\mathscr{P}$ such that both $\mathscr{P}$ and $\sim\mathscr{P}$ lead to contradiction: Assume $\mathscr{P}$ ($\sim\mathscr{P}$), obtain the first contradiction, and conclude $\sim\mathscr{P}$ ($\mathscr{P}$); then obtain the second contradiction — this is the one you want.

Again, the priority order is not the frequency order. The frequency is likely to be something like SC1, SC3, SC4, SC2. Also sometimes, but not always, SC3 and SC4 coincide: in deriving the opposite of some negation, you end up assuming a $\mathscr{P}$ such that $\mathscr{P}$ and $\sim\mathscr{P}$ lead to contradiction.

SC1. *Break accessible formulas down into atomics and negated atomics.* As we have already said, if there is a contradiction to be had, and you can break premises into atomics and negated atomics, the contradiction *will* appear at that level. Thus, for example,

(AK)

1.	$A \wedge B$	P
2.	$\sim B$	P
3.	C	A (c, $\sim$I)
	$\bot$	
	$\sim C$	2-_ $\sim$I

1.	$A \wedge B$	P
2.	$\sim B$	P
3.	C	A (c, $\sim$I)
4.	A	1 $\wedge$E
5.	B	1 $\wedge$E
6.	$\bot$	5,2 $\bot$I
7.	$\sim C$	3-6 $\sim$I

Our strategy for the main goal is SG4 with an application of $\sim$I. Then the aim is to obtain a contradiction. And our first thought is to break accessible lines down to atomics and negated atomics. Perhaps this example is too simple. And you may wonder about the point of getting A at (4) — there *is* no need for A at (4). But this merely illustrates the point: if you can get to atomics and negated atomics ("randomly" as it were) the contradiction will appear in the end.

As another example, try showing $A \wedge (B \wedge \sim C), \sim F \rightarrow D, (A \wedge D) \rightarrow C \vdash_{ND} F$. Here is the completed derivation in two stages.

(AL)

1.	$A \wedge (B \wedge \sim C)$	P	1.	$A \wedge (B \wedge \sim C)$	P
2.	$\sim F \rightarrow D$	P	2.	$\sim F \rightarrow D$	P
3.	$(A \wedge D) \rightarrow C$	P	3.	$(A \wedge D) \rightarrow C$	P
4.	$\quad \sim F$	A (c, $\sim$E)	4.	$\quad \sim F$	A (c, $\sim$E)
			5.	$\quad D$	2,4 $\rightarrow$E
			6.	$\quad A$	1 $\wedge$E
			7.	$\quad A \wedge D$	6,5 $\wedge$I
			8.	$\quad C$	3,7 $\rightarrow$E
			9.	$\quad B \wedge \sim C$	1 $\wedge$E
			10.	$\quad \sim C$	9 $\wedge$E
	$\quad \bot$		11.	$\quad \bot$	8,10 $\bot$I
	F	4-__ $\sim$E	12.	F	4-11 $\sim$E

This time, our strategy for the goal falls through to SG5. After that, again, our goal is to obtain a contradiction — and our first thought is to break premises down to atomics and negated atomics. The assumption $\sim F$ gets us D with (2). We can get A from (1), and then C with the A and D together. Then $\sim C$ follows from (1) by a couple applications of $\wedge$E. You might proceed to get the atomics in a different order, but the basic idea of any such derivation is likely to be the same.

SC2. *Given a disjunction in a subderivation for $\sim$E or $\sim$I, go for $\bot$ by $\vee$E.* This strategy applies only occasionally, though it is related to one that is common for the quantificational case. In most cases, you will have applied $\vee$E by SG2 prior to setting up for $\sim$E or $\sim$I. In some cases, however, a disjunction is "uncovered" only inside a subderivation for a tilde rule. In any such case, SC2 has high priority for the same reasons as SG2: You can only be *better off* in your attempt to reach a contradiction inside the subderivations for $\vee$E than before. So the strategy says to set $\bot$ as the goal you need for $\sim$E or $\sim$I, and go for *it* by $\vee$E.

given

a.	$\mathcal{P}$	A (c, ∼I)
b.	$\mathcal{A} \vee \mathcal{B}$	
	$\bot$	
	$\sim\mathcal{P}$	a-__ ∼I

use

a.	$\mathcal{P}$	A (c, ∼I)
b.	$\mathcal{A} \vee \mathcal{B}$	
c.	$\mathcal{A}$	A (c, b∨E)
d.	$\bot$	
e.	$\mathcal{B}$	A (c, c∨E)
f.	$\bot$	
g.	$\bot$	b,c-d,e-f ∨E
	$\sim\mathcal{P}$	a-g ∼I

Observe that, since the subderivations for ∨E have goal ⊥, they have exit strategy c rather than g. Here is another advantage of our standard use of ⊥. Because ⊥ is a particular sentence, it works as a goal sentence for this rule. We might obtain ⊥ by one contradiction in the first subderivation, and by another in the second. But, once we have obtained ⊥ in each, we are in a position to exit by ∨E in the usual way, and so to apply ∼I.

Here is an example. We show $\sim A \wedge \sim B \vdash_{ND} \sim(A \vee B)$. The derivation is in four stages.

(AM)

1.	$\sim A \wedge \sim B$	P
2.	$A \vee B$	A (c, ∼I)
	$\bot$	
	$\sim(A \vee B)$	2-__ ∼I

1.	$\sim A \wedge \sim B$	P
2.	$A \vee B$	A (c, ∼I)
3.	A	A (c, 2∨E)
	$\bot$	
	B	A (c, 2∨E)
	$\bot$	
	$\bot$	2, __, __ ∨E
	$\sim(A \vee B)$	2-__ ∼I

In this case, our strategy for the goal is SG4. The disjunction appears only inside the subderivation as the assumption for ∼I. We might obtain $\sim A$ and $\sim B$ from (1) but after that, there are no more atomics or negated atomics to be had. So we fall through to SC2, with ⊥ as the goal for ∨E.

1.	$\sim A \wedge \sim B$	P
2.	$\quad A \vee B$	A (c, $\sim$I)
3.	$\quad\quad A$	A (c, 2$\vee$E)
4.	$\quad\quad \sim A$	1 $\wedge$E
5.	$\quad\quad \bot$	3,4 $\bot$I
6.	$\quad\quad B$	A (c, 2$\vee$E)
	$\quad\quad \bot$	
	$\quad \bot$	2,3-5, __ $\vee$E
	$\sim(A \vee B)$	2-__ $\sim$I

1.	$\sim A \wedge \sim B$	P
2.	$\quad A \vee B$	A (c, $\sim$I)
3.	$\quad\quad A$	A (c, 2$\vee$E)
4.	$\quad\quad \sim A$	1 $\wedge$E
5.	$\quad\quad \bot$	3,4 $\bot$I
6.	$\quad\quad B$	A (c, 2$\vee$E)
7.	$\quad\quad \sim B$	1 $\wedge$E
8.	$\quad\quad \bot$	6,7 $\bot$I
9.	$\quad \bot$	2,3-5,6-8 $\vee$E
10.	$\sim(A \vee B)$	2-9 $\sim$I

The first subderivation is easily completed from atomics and negated atomics. And the second is completed the same way. Observe that it is only because of our assumptions for $\vee$E that we are able to get the contradictions at all.

SC3. *Set as goal the opposite of some negation (something that cannot itself be broken down). Then apply standard strategies for the goal.* You will find yourself using this strategy often, after SC1. In the ordinary case, if accessible formulas cannot be broken into atomics and negated atomics, it is because complex forms are "sealed off" by main operator $\sim$. The tilde blocks SC1 or SC2. But you can turn this lemon to lemonade: taking the complex $\sim\mathcal{Q}$ as one half of a contradiction, set $\mathcal{Q}$ as goal. For some complex $\mathcal{Q}$,

given

a.	$\sim\mathcal{Q}$	
b.	$\quad \mathcal{A}$	A (c, $\sim$I)
	$\quad \bot$	
	$\sim\mathcal{A}$	

use

a.	$\sim\mathcal{Q}$	
b.	$\quad \mathcal{A}$	A (c, $\sim$I)
c	$\quad \mathcal{Q}$	(goal)
	$\quad \bot$	c,a $\bot$I
	$\sim\mathcal{A}$	

We are after a contradiction. Supposing that we cannot break $\sim\mathcal{Q}$ into its parts, our efforts to apply other strategies for a contradiction are frustrated. But SC3 offers an alternative: Set $\mathcal{Q}$ itself as a new goal and use this with $\sim\mathcal{Q}$ to reach $\bot$. Then strategies for the new goal take over. If we reach the new goal, we have the contradiction we need.

As an example, try showing $B, \sim(A \rightarrow B) \vdash_{ND} \sim A$. Here is the derivation in four stages.

(AN)

1.	B	P
2.	$\sim(A \to B)$	P
3.	A	A (c, $\sim$I)
	$\bot$	
	$\sim A$	3-_ $\sim$I

1.	B	P
2.	$\sim(A \to B)$	P
3.	A	A (c, $\sim$I)
	$A \to B$	(goal)
	$\bot$	_,2 $\bot$I
	$\sim A$	3-_ $\sim$I

Our strategy for the goal is SG4; for main operator $\sim$ we set up to get the goal by $\sim$I. So we need a contradiction. In this case, there is nothing to be done by way of obtaining atomics and negated atomics, and there is no disjunction. So we fall through to strategy SC3. $\sim(A \to B)$ on (2) has main operator $\sim$, so we set $A \to B$ as a new subgoal with the idea to use it for contradiction.

1.	B	P
2.	$\sim(A \to B)$	P
3.	A	A (c, $\sim$I)
4.	A	A (g, $\to$I)
	B	(goal)
	$A \to B$	4-_ $\to$I
	$\bot$	_,2 $\bot$I
	$\sim A$	3-_ $\sim$I

1.	B	P
2.	$\sim(A \to B)$	P
3.	A	A (c, $\sim$I)
4.	A	A (g, $\to$I)
5.	B	1 R
6.	$A \to B$	4-5 $\to$I
7.	$\bot$	6,2 $\bot$I
8.	$\sim A$	3-7 $\sim$I

Since $A \to B$ is a definite subgoal, we proceed with strategies for the goal in the usual way. The main operator is $\to$ so we set up to get it by $\to$I. The subderivation is particularly easy to complete. And we finish by executing the exit strategies as planned.

SC4. *For some $\mathcal{P}$ such that both $\mathcal{P}$ and $\sim\mathcal{P}$ lead to contradiction*: Assume $\mathcal{P}$ ($\sim\mathcal{P}$), obtain the first contradiction, and conclude $\sim\mathcal{P}$ ($\mathcal{P}$); then obtain the second contradiction — this is the one you want.

given

a.	$\mathcal{A}$	A (c, $\sim$I)
	$\bot$	
	$\sim\mathcal{A}$	

use

a.	$\mathcal{A}$	A (c, $\sim$I)
b.	$\mathcal{P}$	A (c, $\sim$I)
c.	$\bot$	
	$\sim\mathcal{P}$	b-c $\sim$I
d.	$\bot$	
	$\sim\mathcal{A}$	a-d $\sim$I

The essential point is that both $\mathcal{P}$ and $\sim\mathcal{P}$ somehow lead to contradiction. Thus the assumption of one leads by $\sim$I or $\sim$E to the other; and since *both* lead to contradiction, you end up with the contradiction you need. This is often a powerful way of making progress when none seems possible by other means.

Let us try to show $A \leftrightarrow B$, $B \leftrightarrow C$, $C \leftrightarrow \sim A \vdash_{ND} K$. Here is the derivation in four stages.

(AO)

1.	$A \leftrightarrow B$	P
2.	$B \leftrightarrow C$	P
3.	$C \leftrightarrow \sim A$	P
4.	$\quad \sim K$	A (c, $\sim$E)
	$\quad \bot$	
	K	4-_ $\sim$E

1.	$A \leftrightarrow B$	P
2.	$B \leftrightarrow C$	P
3.	$C \leftrightarrow \sim A$	P
4.	$\quad \sim K$	A (c, $\sim$E)
5.	$\quad\quad A$	A (c, $\sim$I)
	$\quad\quad \bot$	
	$\quad \sim A$	5-_ $\sim$I
	$\quad \bot$	
	K	4-_ $\sim$E

Our strategy for the goal falls all the way through to SG5. So we assume the negation of the goal, and go for a contradiction. In this case, there are no atomics or negated atomics to be had. There is no disjunction under the scope of the negation, and no formula is itself a negation such that we could reiterate and build up to the opposite. But given formula A we can use $\leftrightarrow$E to reach $\sim A$ and so contradiction. And, similarly, given $\sim A$ we can use $\leftrightarrow$E to reach A and so contradiction. So, following SC4, we assume one of them to get the other.

1.	$A \leftrightarrow B$	P
2.	$B \leftrightarrow C$	P
3.	$C \leftrightarrow \sim A$	P
4.	$\sim K$	A (c, $\sim$E)
5.	A	A (c, $\sim$I)
6.	B	1,5 $\leftrightarrow$E
7.	C	2,6 $\leftrightarrow$E
8.	$\sim A$	3,7 $\leftrightarrow$E
9.	$\bot$	5,8 $\bot$I
10.	$\sim A$	5-9 $\sim$I
	$\bot$	
	K	4-_ $\sim$E

1.	$A \leftrightarrow B$	P
2.	$B \leftrightarrow C$	P
3.	$C \leftrightarrow \sim A$	P
4.	$\sim K$	A (c, $\sim$E)
5.	A	A (c, $\sim$I)
6.	B	1,5 $\leftrightarrow$E
7.	C	2,6 $\leftrightarrow$E
8.	$\sim A$	3,7 $\leftrightarrow$E
9.	$\bot$	5,8 $\bot$I
10.	$\sim A$	5-9 $\sim$I
11.	C	3,10 $\leftrightarrow$E
12.	B	2,11 $\leftrightarrow$E
13.	A	1,12 $\leftrightarrow$E
14.	$\bot$	13,10 $\bot$I
15.	K	4-14 $\sim$E

The first contradiction appears easily at the level of atomics and negated atomics. This gives us $\sim A$. And with $\sim A$, the second contradiction also comes easily, at the level of atomics and negated atomics.

Though it can be useful, this strategy is often difficult to see. And there is no obvious way to give a strategy for using the strategy! The best thing to say is that you should *look for it* when the other strategies seem to fail.

Let us consider an extended example which combines some of the strategies. We show that $\sim A \rightarrow B \vdash_{ND} B \vee A$.

(AP)

1.	$\sim A \rightarrow B$	P
	$B \vee A$	

In this case, we do not see a contradiction in the premises; there is no formula with main operator $\vee$ in the premises; and the goal does not appear in the premises. So we might try going for the goal by $\vee$I in application of SG4. This would require getting a B or an A. It is reasonable to go this way, but it turns out to be a dead end. (You should convince yourself that this is so.) Thus we fall through to SG5.

1.	$\sim A \rightarrow B$	P
2.	$\sim(B \vee A)$	A (c, $\sim$E)
	$\bot$	
	$B \vee A$	2-_ $\sim$E

Especially considering our goal has main operator $\vee$, set up to get the goal by $\sim$E.

To get a contradiction, our first thought is to go for atomics and negated atomics. But there is nothing to be done. Similarly, there is no formula with main operator $\vee$. So we fall through to SC3 and continue as follows.

1.	$\sim A \rightarrow B$	P	
2.	$\quad\sim(B \vee A)$	A (c, $\sim$E)	Given a negation that cannot be broken down, set up to get the contradiction by building up to the opposite.
	$\quad B \vee A$		
	$\quad\bot$	_,2 $\bot$I	
	$B \vee A$	2-_ $\sim$E	

It might seem that we have made no progress, since our new goal is no different than the original! But there is progress insofar as we have a premise not available before (more on this in a moment). At this stage, we *can* get the goal by $\vee$I. Either side will work, but it is easier to start with the A. So we set up for that.

1.	$\sim A \rightarrow B$	P	
2.	$\quad\sim(B \vee A)$	A (c, $\sim$E)	For a goal with main operator $\vee$, go for the goal by $\vee$I
	$\quad A$		
	$\quad B \vee A$	_ $\vee$I	
	$\quad\bot$	_,2 $\bot$I	
	$B \vee A$	2-_ $\sim$E	

Now the goal is atomic. Again, there is no contradiction or formula with main operator $\vee$ in the premises. The goal is not in the premises in any form we can hope to exploit. And the goal has no main operator. So, again, we fall through to SG5.

1.	$\sim A \rightarrow B$	P	
2.	$\quad\sim(B \vee A)$	A (c, $\sim$E)	
3.	$\quad\quad\sim A$	A (c, $\sim$E)	Especially for atomics, go for the goal by $\sim$E
	$\quad\quad\bot$		
	$\quad A$	3-_ $\sim$E	
	$\quad B \vee A$	_ $\vee$I	
	$\quad\bot$	_,2 $\bot$I	
	$B \vee A$	2-_ $\sim$E	

Again, our first thought is to get atomics and negated atomics. We can get B from lines (1) and (3) by $\rightarrow$E. But that is all. So we will not get a contradiction from atomics and negated atomics alone. There is no formula with main operator $\vee$. However, the possibility of getting a B suggests that we *can* build up to the opposite of line (2). That is, we complete the subderivation as follows, and follow our exit strategies to complete the whole.

1.	$\sim A \rightarrow B$	P	
2.	$\quad \sim(B \vee A)$	A (c, $\sim$E)	
3.	$\quad\quad \sim A$	A (c, $\sim$E)	
4.	$\quad\quad B$	1,3 $\rightarrow$E	Get the contradiction by building up to the opposite of an existing negation.
5.	$\quad\quad B \vee A$	4 $\vee$I	
6.	$\quad\quad \bot$	5,2 $\bot$I	
7.	$\quad A$	3-6 $\sim$E	
8.	$\quad B \vee A$	7 $\vee$I	
9.	$\quad \bot$	8,2 $\bot$I	
10.	$B \vee A$	2-9 $\sim$E	

A couple of comments: First, observe that we build up to the opposite of $\sim(B \vee A)$ *twice*, coming at it from different directions. First we obtain the left side B and use $\vee$I to obtain the whole, then the right side A and use $\vee$I to obtain the whole. This is typical with negated disjunctions. Second, note that this derivation might be reconceived as an instance of SC4. $\sim A$ gets us B, and so $B \vee A$, which contradicts $\sim(B \vee A)$. But A gets us $B \vee A$ which, again, contradicts $\sim(B \vee A)$. So both A and $\sim A$ lead to contradiction; so we assume one ($\sim A$), and get the first contradiction; this gets us A, from which the second contradiction follows.

The general pattern of this derivation is typical for formulas with main operator $\vee$ in *ND*. For $\mathcal{P} \vee \mathcal{Q}$ we may not be able to prove either $\mathcal{P}$ or $\mathcal{Q}$ from scratch — so that the formula is not directly provable by $\vee$I. However, it may be *indirectly* provable. If it is provable at all, it *must* be that the negation of one side forces the other. So it must be possible to get the $\mathcal{P}$ or the $\mathcal{Q}$ under the *additional* assumption that the other is false. This makes possible an argument of the following form.

	a.	$\quad \sim(\mathcal{P} \vee \mathcal{Q})$	A (c, $\sim$E
	b.	$\quad\quad \sim\mathcal{P}$	A (c, $\sim$E)
		$\quad\quad \vdots$	
	c.	$\quad\quad \mathcal{Q}$	
(AQ)	d.	$\quad\quad \mathcal{P} \vee \mathcal{Q}$	c $\vee$I
	e.	$\quad\quad \bot$	d,a $\bot$I
	f.	$\quad \mathcal{P}$	b-e $\sim$E
	g.	$\quad \mathcal{P} \vee \mathcal{Q}$	f $\vee$I
	h.	$\quad \bot$	g,a $\bot$I
	i.	$\mathcal{P} \vee \mathcal{Q}$	a-h $\sim$E

The "work" in this routine is getting from the negation of one side of the disjunction to the other. Thus if from the assumption $\sim\mathcal{P}$ it is possible to derive $\mathcal{Q}$, all the rest is automatic! We have just seen an extended example (AP) of this pattern. It may be seen as an application of SC3 or SC4 (or both). Where a disjunction may be provable but not provable by $\vee$I, it *will* work by this method! So in difficult cases when the goal is a disjunction, it is wise to think about whether you can get one side from the negation of the other. If you can, set up as above. (And reconsider this method, when we get to a simplified version in the extended system *ND+*).

This example was fairly difficult! You may see some longer, but you will not see many harder. The strategies are not a cookbook for performing all derivations — doing derivations remains an art. But the strategies will give you a good start, and take you a long way through the exercises that follow. The theorems immediately below again foreshadow rules of *ND+*.

*T6.21. $\vdash_{ND} \sim(\mathcal{A} \wedge \mathcal{B}) \leftrightarrow (\sim\mathcal{A} \vee \sim\mathcal{B})$

T6.22. $\vdash_{ND} \sim(\mathcal{A} \vee \mathcal{B}) \leftrightarrow (\sim\mathcal{A} \wedge \sim\mathcal{B})$

T6.23. $\vdash_{ND} (\sim\mathcal{A} \rightarrow \mathcal{B}) \leftrightarrow (\mathcal{A} \vee \mathcal{B})$

T6.24. $\vdash_{ND} (\mathcal{A} \rightarrow \mathcal{B}) \leftrightarrow (\sim\mathcal{A} \vee \mathcal{B})$

T6.25. $\vdash_{ND} [\mathcal{A} \wedge (\mathcal{B} \vee \mathcal{C})] \leftrightarrow [(\mathcal{A} \wedge \mathcal{B}) \vee (\mathcal{A} \wedge \mathcal{C})]$

T6.26. $\vdash_{ND} [\mathcal{A} \vee (\mathcal{B} \wedge \mathcal{C})] \leftrightarrow [(\mathcal{A} \vee \mathcal{B}) \wedge (\mathcal{A} \vee \mathcal{C})]$

T6.27. $\vdash_{ND} (\mathcal{A} \leftrightarrow \mathcal{B}) \leftrightarrow [(\mathcal{A} \rightarrow \mathcal{B}) \wedge (\mathcal{B} \rightarrow \mathcal{A})]$

T6.28. $\vdash_{ND} (\mathcal{A} \leftrightarrow \mathcal{B}) \leftrightarrow [(\mathcal{A} \wedge \mathcal{B}) \vee (\sim\mathcal{A} \wedge \sim\mathcal{B})]$

T6.29. $\vdash_{ND} [\mathcal{A} \leftrightarrow (\mathcal{B} \leftrightarrow \mathcal{C})] \leftrightarrow [(\mathcal{A} \leftrightarrow \mathcal{B}) \leftrightarrow \mathcal{C}]$

E6.17. Each of the following begins with a simple application of ~I or ~E for SG4 or SG5. Complete the derivations, and *explain* your use of strategies for a contradiction. Hint: Each of the strategies for a contradiction is used at least once.

*a.
1. $A \wedge B$ P
2. $\sim(A \wedge C)$ P
3. C A (c, ~I)

$\bot$

$\sim C$

b.
1. $(\sim B \vee \sim A) \rightarrow D$ P
2. $C \wedge \sim D$ P
3. $\sim B$ A (c, ~E)

$\bot$

B

c.
1. $A \wedge B$ P
2. $\sim A \vee \sim B$ A (c, ~I)

$\bot$

$\sim(\sim A \vee \sim B)$

d.
1. $A \leftrightarrow \sim A$ P
2. B A (c, ~I)

$\bot$

$\sim B$

e.
1. $\sim(A \rightarrow B)$ P
2. $\sim A$ A (c, ~E)

$\bot$

A

E6.18. Produce derivations to show each of the following. No worked out answers are provided. However, if you get stuck, you will find strategy hints in the back.

*a. $A \to \sim(B \wedge C),\ B \to C \vdash_{ND} A \to \sim B$

*b. $\vdash_{ND} \sim(A \to A) \to A$

*c. $A \vee B \vdash_{ND} \sim(\sim A \wedge \sim B)$

*d. $\sim(A \wedge B),\ \sim(A \wedge \sim B) \vdash_{ND} \sim A$

*e. $\vdash_{ND} A \vee \sim A$

*f. $\vdash_{ND} A \vee (A \to B)$

*g. $A \vee \sim B,\ \sim A \vee \sim B \vdash_{ND} \sim B$

*h. $A \leftrightarrow (\sim B \vee C),\ B \to C \vdash_{ND} A$

*i. $A \leftrightarrow B \vdash_{ND} (C \leftrightarrow A) \leftrightarrow (C \leftrightarrow B)$

*j. $A \leftrightarrow \sim(B \leftrightarrow \sim C),\ \sim(A \vee B) \vdash_{ND} C$

*k. $[C \vee (A \vee B)] \wedge (C \to E),\ A \to D,\ D \to \sim A \vdash_{ND} C \vee B$

*l. $\sim(A \to B),\ \sim(B \to C) \vdash_{ND} \sim D$

*m. $C \to \sim A,\ \sim(B \wedge C) \vdash_{ND} (A \vee B) \to \sim C$

*n. $\sim(A \leftrightarrow B) \vdash_{ND} \sim A \leftrightarrow B$

*o. $A \leftrightarrow B,\ B \leftrightarrow \sim C \vdash_{ND} \sim(A \leftrightarrow C)$

*p. $A \vee B,\ \sim B \vee C,\ \sim C \vdash_{ND} A$

*q. $(\sim A \vee C) \vee D,\ D \to \sim B \vdash_{ND} (A \wedge B) \to C$

*r. $A \vee D,\ \sim D \leftrightarrow (E \vee C),\ (C \wedge B) \vee [C \wedge (F \to C)] \vdash_{ND} A$

*s. $(A \vee B) \vee (C \wedge D),\ (A \leftrightarrow E) \wedge (B \to F),\ G \leftrightarrow \sim(E \vee F),\ C \to B \vdash_{ND} \sim G$

*t. $(A \vee B) \wedge \sim C,\ \sim C \to (D \wedge \sim A),\ B \to (A \vee E) \vdash_{ND} E \vee F$

*E6.19. Produce derivations to demonstrate each of T6.21 - T6.28. It turns out that T6.29 is particularly challenging. Note that its demonstration (from left-to-right) is left for E6.20e.

E6.20. Produce derivations to show each of the following. These are particularly challenging. If you can get them, you are doing very well! (In keeping with the spirit of the challenge, no help is provided in the back of the book.)

a. $(A \vee B) \rightarrow (A \vee C) \vdash_{ND} A \vee (B \rightarrow C)$

b. $A \rightarrow (B \vee C) \vdash_{ND} (A \rightarrow B) \vee (A \rightarrow C)$

c. $(A \leftrightarrow B) \leftrightarrow (C \leftrightarrow D) \vdash_{ND} (A \leftrightarrow C) \rightarrow (B \rightarrow D)$

d. $\sim(A \leftrightarrow B), \sim(B \leftrightarrow C), \sim(C \leftrightarrow A) \vdash_{ND} \sim K$

e. $A \leftrightarrow (B \leftrightarrow C) \vdash_{ND} (A \leftrightarrow B) \leftrightarrow C$

E6.21. For each of the following, produce a good translation including interpretation function. Then use a derivation to show that the argument is valid in *ND*. The first two are suggested from the history of philosophy; the last is our familiar case from p. 2.

a. We have knowledge about numbers.
If Platonism is true, then numbers are not in spacetime.
Either numbers are in spacetime, or we do not interact with them.
We have knowledge about numbers only if we interact with them.
———
Platonism is not true.

b. There is evil.
If god is good, then there is no evil unless he has morally sufficient reasons for allowing it.
If god is both omnipotent and omniscient, then he does not have morally sufficient reasons for allowing evil.
———
God is not good, omnipotent and omniscient.

c. If Bob goes to the fair, then so do Daniel and Edward. Albert goes to the fair only if Bob or Carol go. If Daniel goes, then Edward goes only if Fred goes. But not both Fred and Albert go. So Albert goes to the fair only if Carol goes too.

d. If I think dogs fly, then I am insane or they have really big ears. But if dogs do not have really big ears, then I am not insane. So either I do not think dogs fly, or they have really big ears.

e. If the maid did it, then it was done with a revolver only if it was done in the parlor. But if the butler is innocent, then the maid did it unless it was done in the parlor. The maid did it only if it was done with a revolver, while the butler is guilty if it did happen in the parlor. So the butler is guilty.

E6.22. For each of the following concepts, explain in an essay of about two pages, so that (high-school age) Hannah could understand. In your essay, you should (i) identify the objects to which the concept applies, (ii) give and explain the definition, and give and explicate examples of your own construction (iii) where the concept applies, and (iv) where it does not. Your essay should exhibit an understanding of methods from the text.

a. Derivations as games, and the condition on rules.

b. Accessibility, and auxiliary assumptions.

c. The rules $\vee$I and $\vee$E.

d. The strategies for a goal.

e. The strategies for a contradiction.

6.3 Quantificational

Part II

Transition: Reasoning About Logic

Introductory

We have expended a great deal of energy learning to do logic. What we have learned constitutes the complete classical predicate calculus with equality. This is a system of tremendous power including for reasoning in foundations of arithmetic.

But our work itself raises questions. In chapter 4 we used truth trees and tables for an account of the conditions under which sentential formulas are true and arguments are valid. In the quantificational case, though, we were not able to use our graphical methods for a general account of truth and validity — there were simply too many branches, and too many interpretations, for a general account by means of trees. Thus there is an open question about whether and how quantificational validity can be shown.

And once we have introduced our notions of validity, many interesting questions can be asked about how they work: are the arguments that are valid in *AD* the same as the ones that are valid in *ND*? are the arguments that are valid in *ND* the same as the ones that are quantificationally valid? Are the theorems of Q the same as the theorems of PA? are theorems of PA the same as the truths on N the standard interpretation for number theory? Is it possible for a computing device to identify the theorems of the different logical systems?

It is one thing to ask such questions, and perhaps amazing that there are demonstrable answers. We will come to that. However, in this short section we do not attempt answers. Rather, we put ourselves in a position to think about answers by introducing methods for thinking about logic. Thus this part looks both backward and forward: By our methods we plug the hole left from chapter 4: in chapter 7 we accomplish what could not be done with the tables and trees of chapter 4, and are able to demonstrate quantificational validity. At the same time, we lay a foundation to ask and answer core questions about logic.

Chapter 7 begins with our basic method of reasoning from definitions. Chapter **??** introduces mathematical induction. These methods are important not only for results, but for their own sakes, as part of the "package" that comes with mathematical logic.

Chapter 7

Direct Semantic Reasoning

It is the task of this chapter to think about reasoning directly from definitions. Frequently, students who already reason quite skillfully with definitions flounder when asked to do so explicitly, in the style of this chapter.[1] Thus I propose to begin in a restricted context — one with which we are already familiar, using a fairly rigid framework as a guide. Perhaps you first learned to ride a bicycle with training wheels, but eventually learned to ride without them, and so to go faster, and to places other than the wheels would let you go. Similarly, in the end, we will want to apply our methods beyond the restricted context in which we begin, working outside the initial framework. But the framework should give us a good start. In this section, then, I introduce the framework in the context of reasoning for specifically *semantic* notions, and against the background of semantic reasoning we have already done.

In chapter 4 we used truth trees and tables for an account of the conditions under which sentential formulas are true and arguments are valid. In the quantificational case, though, we were not able to use our graphical methods for a general account of truth and validity — there were simply too many branches, and too many interpretations, for a general account by means of trees. For a complete account, we will need to reason more directly from the definitions. But the tables and trees *do* exhibit the semantic definitions. So we can build on what we have already done with them. Our goal will be to move past the tables and trees, and learn to function without them. After some

[1] The ability to reason clearly and directly with definitions is important not only here, but also beyond. In philosophy, compare the humorous, but also serious, verb *to chisholm* after Roderick Chisholm, who was a master of the technique — where one proposes a definition; considers a counterexample; modifies to account for the example; considers another counterexample; modifies again; and so forth. As, "He started with definition (d.8) and kept chisholming away at it until he ended up with (d.8$''''''''$)" (*The Philosopher's Lexicon*). Such reasoning is impossible to understand apart from explicit attention to consequences of definitions of the sort we have in mind.

general remarks, we start with the sentential case, and move to the quantificational.

7.1 General

I begin with some considerations about what we are trying to accomplish, and how it is related to what we have done. Consider the following row of a truth table, meant to show that $B \to C \nvDash_s \sim B$.

(A)

B	C	$B \to C$	/	$\sim B$
T	T	T ***T*** T		***F*** T

Since there is an interpretation on which the premise is true and the conclusion is not, the argument is not sententially valid. Now, what justifies the move from $\mathsf{I}[B] = \mathsf{T}$ and $\mathsf{I}[C] = \mathsf{T}$, to the conclusion that $B \to C$ is T? One might respond, "the truth table." But the truth table, T($\to$) is itself derived from definition ST($\to$). According to ST($\to$), for sentences $\mathcal{P}$ and $\mathcal{Q}$, $\mathsf{I}[(\mathcal{P} \to \mathcal{Q})] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{P}] = \mathsf{F}$ or $\mathsf{I}[\mathcal{Q}] = \mathsf{T}$ (or both). In this case, $\mathsf{I}[C] = \mathsf{T}$; so $\mathsf{I}[B] = \mathsf{F}$ or $\mathsf{I}[C] = \mathsf{T}$; so the condition from ST($\to$) is met, and $\mathsf{I}[B \to C] = \mathsf{T}$. It may seem odd to move from $\mathsf{I}[C] = \mathsf{T}$; to $\mathsf{I}[B] = \mathsf{F}$ or $\mathsf{I}[C] = \mathsf{T}$, when in fact $\mathsf{I}[B] = \mathsf{T}$; but it is certainly *correct* — just as for $\vee$I in *ND*, the point is merely to make explicit that, in virtue of the fact that $\mathsf{I}[C] = \mathsf{T}$, the interpretation meets the disjunctive condition from ST($\to$). And what justifies the move from $\mathsf{I}[B] = \mathsf{T}$ to the conclusion that $\mathsf{I}[\sim B] = \mathsf{F}$? ST($\sim$). According to ST($\sim$), for any sentence $\mathcal{P}$, $\mathsf{I}[\sim\mathcal{P}] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{P}] = \mathsf{F}$. In this case, $\mathsf{I}[B] = \mathsf{T}$; and since $\mathsf{I}[B]$ is not F, $\mathsf{I}[\sim B]$ is not T; so $\mathsf{I}[\sim B] = \mathsf{F}$. Similarly, definition SV justifies the conclusion that the argument is not sententially valid. According to SV, $\Gamma \vDash_s \mathcal{P}$ just in case there is no sentential interpretation I such that $\mathsf{I}[\Gamma] = \mathsf{T}$ but $\mathsf{I}[\mathcal{P}] = \mathsf{F}$. Since we have produced an I such that $\mathsf{I}[B \to C] = \mathsf{T}$ but $\mathsf{I}[\sim B] = \mathsf{F}$, it follows that $B \to C \nvDash_s \sim B$. So the definitions drive the tables.

In chapter 4, we used tables to express these conditions. But we *might* have reasoned directly.

(B) Consider any interpretation I such that $\mathsf{I}[B] = \mathsf{T}$ and $\mathsf{I}[C] = \mathsf{T}$. Since $\mathsf{I}[C] = \mathsf{T}$, $\mathsf{I}[B] = \mathsf{F}$ or $\mathsf{I}[C] = \mathsf{T}$; so by ST($\to$), $\mathsf{I}[B \to C] = \mathsf{T}$. But since $\mathsf{I}[B] = \mathsf{T}$, by ST($\sim$), $\mathsf{I}[\sim B] = \mathsf{F}$. So there is a sentential interpretation I such that $\mathsf{I}[B \to C] = \mathsf{T}$ but $\mathsf{I}[\sim B] = \mathsf{F}$; so by SV, $B \to C \nvDash_s \sim B$.

Presumably, all this is "contained" in the one line of the truth table, when we use it to conclude that the argument is not sententially valid.

Similarly, consider the following table, meant to show that $\sim\sim A \vDash_s \sim A \to A$.

(C)

A	$\sim$	$\sim$	A	/	$\sim$	A	$\to$	A
T	***T***	F	T		F	T	***T***	T
F	***F***	T	F		T	F	***F***	F

Since there is no row where the premise is true and the conclusion is false, the argument is sententially valid. Again, ST($\sim$) and ST($\to$) justify the way you build the table. And SV lets you conclude that the argument is sententially valid. Since no row makes the premise true and the conclusion false, and any sentential interpretation is like some row in its assignment to A, no sentential interpretation makes the premise true and conclusion false; so, by SV, the argument is sententially valid.

Thus the table represents reasoning as follows (omitting the second row). To follow, notice how we simply reason through each "place" in a row, and then about whether the row shows invalidity.

(D) For any sentential interpretation I, either (i) $\mathsf{I}[A] = \mathsf{T}$ or (ii) $\mathsf{I}[A] = \mathsf{F}$. Suppose (i); then $\mathsf{I}[A] = \mathsf{T}$; so by ST($\sim$), $\mathsf{I}[\sim A] = \mathsf{F}$; so by ST($\sim$) again, $\mathsf{I}[\sim\sim A] = \mathsf{T}$. But $\mathsf{I}[A] = \mathsf{T}$, and by ST($\sim$), $\mathsf{I}[\sim A] = \mathsf{F}$; from either of these it follows that $\mathsf{I}[\sim A] = \mathsf{F}$ or $\mathsf{I}[A] = \mathsf{T}$; so by ST($\to$), $\mathsf{I}[\sim A \to A] = \mathsf{T}$. From this either $\mathsf{I}[\sim\sim A] = \mathsf{F}$ or $\mathsf{I}[\sim A \to A] = \mathsf{T}$; so it is not the case that $\mathsf{I}[\sim\sim A] = \mathsf{T}$ and $\mathsf{I}[\sim A \to A] = \mathsf{F}$. Suppose (ii); then by related reasoning... it is not the case that $\mathsf{I}[\sim\sim A] = \mathsf{T}$ and $\mathsf{I}[\sim A \to A] = \mathsf{F}$. So no interpretation makes it the case that $\mathsf{I}[\sim\sim A] = \mathsf{T}$ and $\mathsf{I}[\sim A \to A] = \mathsf{F}$. So by SV, $\sim\sim A \vDash_s \sim A \to A$.

Thus we might recapitulate reasoning in the table. Perhaps we typically "whip through" tables without explicitly considering all the definitions involved. But the definitions *are* involved when we complete the table.

Strictly, though, not all of this is necessary for the conclusion that the argument is valid. Thus, for example, in the reasoning at (i), for the conditional there is no need to establish that both $\mathsf{I}[\sim A] = \mathsf{F}$ and that $\mathsf{I}[A] = \mathsf{T}$. From either, it follows that $\mathsf{I}[\sim A] = \mathsf{F}$ or $\mathsf{I}[A] = \mathsf{T}$; and so by ST($\to$) that $\mathsf{I}[\sim A \to A] = \mathsf{T}$. So we might have omitted one or the other. Similarly at (i) there is no need to make the point that $\mathsf{I}[\sim\sim A] = \mathsf{T}$. What matters is that $\mathsf{I}[\sim A \to A] = \mathsf{T}$, so that $\mathsf{I}[\sim\sim A] = \mathsf{F}$ or $\mathsf{I}[\sim A \to A] = \mathsf{T}$, and it is therefore not the case that $\mathsf{I}[\sim\sim A] = \mathsf{T}$ and $\mathsf{I}[\sim A \to A] = \mathsf{F}$. So reasoning for the full table might be "shortcut" as follows.

(E) For any sentential interpretation either (i) $\mathsf{I}[A] = \mathsf{T}$ or (ii) $\mathsf{I}[A] = \mathsf{F}$. Suppose (i); then $\mathsf{I}[A] = \mathsf{T}$; so $\mathsf{I}[\sim A] = \mathsf{F}$ or $\mathsf{I}[A] = \mathsf{T}$; so by ST($\to$), $\mathsf{I}[\sim A \to A] = \mathsf{T}$. From this either $\mathsf{I}[\sim\sim A] = \mathsf{F}$ or $\mathsf{I}[\sim A \to A] = \mathsf{T}$; so it is not the case that $\mathsf{I}[\sim\sim A] = \mathsf{T}$ and $\mathsf{I}[\sim A \to A] = \mathsf{F}$. Suppose (ii); then $\mathsf{I}[A] = \mathsf{F}$; so by ST($\sim$), $\mathsf{I}[\sim A] = \mathsf{T}$; so by ST($\sim$) again, $\mathsf{I}[\sim\sim A] = \mathsf{F}$; so either $\mathsf{I}[\sim\sim A] = \mathsf{F}$ or $\mathsf{I}[\sim A \to A] = \mathsf{T}$; so it is not the case that $\mathsf{I}[\sim\sim A] = \mathsf{T}$ and $\mathsf{I}[\sim A \to A] = \mathsf{F}$. So no interpretation makes it the case that $\mathsf{I}[\sim\sim A] = \mathsf{T}$ and $\mathsf{I}[\sim A \to A] = \mathsf{F}$. So by SV, $\sim\sim A \vDash_s \sim A \to A$.

This is better. These shortcuts may reflect what you have already done when you realize that, say, a true conclusion eliminates the need to think about the premises on some row of a table. Though the shortcuts make things better, however, the idea of reasoning in this way corresponding to a 4, 8 or more (!) row table remains painful. But there is a way out.

Recall what happens when you apply the short truth-table method from chapter 4 to valid arguments. You start with the assumption that the premises are true and the conclusion is not. If the argument is valid, you reach some conflict so that it is not, in fact, possible to complete the row. Then, as we said on p. 75, you know "in your heart" that the argument is valid. Let us turn this into an official argument form.

(F) Suppose $\sim\sim A \nvDash_s \sim A \rightarrow A$; then by SV, there is an I such that $\mathsf{I}[\sim\sim A] = \mathsf{T}$ and $\mathsf{I}[\sim A \rightarrow A] = \mathsf{F}$. From the former, by ST($\sim$), $\mathsf{I}[\sim A] = \mathsf{F}$. But from the latter, by ST($\rightarrow$), $\mathsf{I}[\sim A] = \mathsf{T}$ and $\mathsf{I}[A] = \mathsf{F}$; and since $\mathsf{I}[\sim A] = \mathsf{T}$, $\mathsf{I}[\sim A] \neq \mathsf{F}$. This is impossible; reject the assumption: $\sim\sim A \vDash_s \sim A \rightarrow A$.

This is better still. The assumption that the argument is invalid leads to the conclusion that for some I, $\mathsf{I}[\sim A] = \mathsf{T}$ and $\mathsf{I}[\sim A] = \mathsf{F}$; but a formula is T just in case it is not F, so this is impossible and we reject the assumption. The pattern is like $\sim$E in *ND*. This approach is particularly important insofar as we do not reason individually about each of the possible interpretations. This is nice in the sentential case, when there are too many to reason about conveniently. And in the quantificational case, we will not be *able* to argue individually about each of the possible interpretations. So we need to avoid talking about interpretations one-by-one.

Thus we arrive at two strategies: To show that an argument is invalid, we produce an interpretation, and show by the definitions that it makes the premises true and the conclusion not. That is what we did in (B) above. To show that an argument is valid, we assume the opposite, and show by the definitions that the assumption leads to contradiction. Again, that is what we did just above, at (F).

Before we get to the details, let us consider an important point about what we are trying to do: Our *reasoning* takes place in the metalanguage, based on the definitions — where object-level expressions are *uninterpreted* apart from the definitions. To see this, ask yourself whether a sentence $\mathcal{P}$ conflicts with $\mathcal{P} \mid \mathcal{P}$. "Well," you might respond, "I have never encountered this symbol '$\mid$' before, so I am not in a position to say." But that is the point: whether $\mathcal{P}$ conflicts with $\mathcal{P} \mid \mathcal{P}$ depends entirely on a definition for *stroke* '$\mid$'. As it happens, this symbol is typically read "not-both" as given by what might be a further clause of ST,

ST($\mid$) For any sentences $\mathcal{P}$ and $\mathcal{Q}$, $\mathsf{I}[(\mathcal{P} \mid \mathcal{Q})] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{P}] = \mathsf{F}$ or $\mathsf{I}[\mathcal{Q}] = \mathsf{F}$ (or both); otherwise $\mathsf{I}[(\mathcal{P} \mid \mathcal{Q})] = \mathsf{F}$.

The resultant table is,

T(|)

$\mathscr{P}$	$\mathscr{Q}$	$\mathscr{P} \mid \mathscr{Q}$
T	T	***F***
T	F	***T***
F	T	***T***
F	F	***T***

$\mathscr{P} \mid \mathscr{Q}$ is false when $\mathscr{P}$ and $\mathscr{Q}$ are both T, and otherwise true. Given this, $\mathscr{P}$ does conflict with $\mathscr{P} \mid \mathscr{P}$. Suppose $\mathsf{I}[\mathscr{P}] = \mathsf{T}$ and $\mathsf{I}[\mathscr{P} \mid \mathscr{P}] = \mathsf{T}$; from the latter, by ST(|), $\mathsf{I}[\mathscr{P}] = \mathsf{F}$ or $\mathsf{I}[\mathscr{P}] = \mathsf{F}$; either way, $\mathsf{I}[\mathscr{P}] = \mathsf{F}$; but this is impossible given our assumption that $\mathsf{I}[\mathscr{P}] = \mathsf{T}$. In fact, $\mathscr{P} \mid \mathscr{P}$ has the same table as $\sim\mathscr{P}$, and $\mathscr{P} \mid (\mathscr{Q} \mid \mathscr{Q})$ the same as $\mathscr{P} \rightarrow \mathscr{Q}$.

(G)

$\mathscr{P}$	$\mathscr{P} \mid \mathscr{P}$
T	***F***
F	***T***

$\mathscr{P}$	$\mathscr{Q}$	$\mathscr{P}$ |	$(\mathscr{Q} \mid \mathscr{Q})$
T	T	***T***	F
T	F	***F***	T
F	T	***T***	F
F	F	***T***	T

From this, we *might* have treated $\sim$ and $\rightarrow$, and so $\wedge$, $\vee$ and $\leftrightarrow$, all as abbreviations for expressions whose only operator is |. At best, however, this leaves official expressions difficult to read. Here is the point that matters: Operators have their significance entirely from the definitions. In this chapter, we make metalinguistic claims *about* object expressions, where these can only be based on the definitions. $\mathscr{P}$ and $\mathscr{P} \mid \mathscr{P}$ do not themselves conflict, apart from the definition which makes $\mathscr{P}$ with $\mathscr{P} \mid \mathscr{P}$ have the consequence that $\mathsf{I}[\mathscr{P}] = \mathsf{T}$ and $\mathsf{I}[\mathscr{P}] = \mathsf{F}$. And similarly for operators with which we are more familiar. At every stage, it is the *definitions* which justify conclusions.

7.2 Sentential

With this much said, it remains possible to become confused about details while working with the definitions. It is one thing to be able to follow such reasoning — as I hope you have been able to do — and another to produce it. The idea now is to make use of something at which we are already good, doing derivations, to further structure and guide the way we proceed. The result will be a sort of derivation system for reasoning about definitions. We build up this system in stages.

7.2.1 Truth

Let us begin with some notation. Where the script characters $\mathscr{A}, \mathscr{B}, \mathscr{C}, \mathscr{D} \ldots$ represent object expressions in the usual way, let the Fraktur characters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \ldots$ represent *metalinguistic* expressions ('$\mathfrak{A}$' is the Fraktur 'A'). Thus $\mathfrak{A}$ might represent an expression of the sort $\mathsf{I}[B] = \mathsf{T}$. Then $\Rightarrow$ and $\Leftrightarrow$ are the metalinguistic conditional and biconditional respectively; $\neg$, $\vartriangle$, $\triangledown$ and $\perp$ represent metalinguistic negation, conjunction, disjunction and contradiction. In practice, negation is indicated by the slash ($\not\models$) as well.

Now consider the following restatement of definition ST. Each clause is given in both a positive and a negative form. For any sentences $\mathcal{P}$ and $\mathcal{Q}$ and interpretation I,

ST	$(\sim)$	$\mathsf{I}[\sim\mathcal{P}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] \neq \mathsf{T}$	$\mathsf{I}[\sim\mathcal{P}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T}$
	$(\rightarrow)$	$\mathsf{I}[\mathcal{P} \rightarrow \mathcal{Q}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] \neq \mathsf{T} \triangledown \mathsf{I}[\mathcal{Q}] = \mathsf{T}$	$\mathsf{I}[\mathcal{P} \rightarrow \mathcal{Q}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}$

Given the new symbols, and that a sentence is F iff it is not true, this is a simple restatement of ST. As we develop our formal system, we will treat the metalinguistic biconditionals both as (replacement) rules and as axioms. Thus, for example, it will be legitimate to move by ST$(\sim)$ directly from $\mathsf{I}[\mathcal{P}] \neq \mathsf{T}$ to $\mathsf{I}[\sim\mathcal{P}] = \mathsf{T}$, moving from right-to-left across the arrow. And similarly in the other direction. Alternatively, it will be appropriate to assert by ST$(\sim)$ the entire biconditional, that $\mathsf{I}[\sim\mathcal{P}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] \neq \mathsf{T}$. For now, we will mostly use the biconditionals, in the first form, as rules.

To manipulate the definitions, we require some rules. These are like ones you have seen before, only pitched at the metalinguistic level.

com $\quad \mathfrak{A} \triangledown \mathfrak{B} \Leftrightarrow \mathfrak{B} \triangledown \mathfrak{A} \qquad \mathfrak{A} \vartriangle \mathfrak{B} \Leftrightarrow \mathfrak{B} \vartriangle \mathfrak{A}$

idm $\quad \mathfrak{A} \Leftrightarrow \mathfrak{A} \triangledown \mathfrak{A} \qquad \mathfrak{A} \Leftrightarrow \mathfrak{A} \vartriangle \mathfrak{A}$

dem $\quad \neg(\mathfrak{A} \vartriangle \mathfrak{B}) \Leftrightarrow \neg\mathfrak{A} \triangledown \neg\mathfrak{B} \qquad \neg(\mathfrak{A} \triangledown \mathfrak{B}) \Leftrightarrow \neg\mathfrak{A} \vartriangle \neg\mathfrak{B}$

cnj $\quad \dfrac{\mathfrak{A}, \mathfrak{B}}{\mathfrak{A} \vartriangle \mathfrak{B}} \qquad \dfrac{\mathfrak{A} \vartriangle \mathfrak{B}}{\mathfrak{A}} \qquad \dfrac{\mathfrak{A} \vartriangle \mathfrak{B}}{\mathfrak{B}}$

dsj $\quad \dfrac{\mathfrak{A}}{\mathfrak{A} \triangledown \mathfrak{B}} \qquad \dfrac{\mathfrak{B}}{\mathfrak{A} \triangledown \mathfrak{B}} \qquad \dfrac{\mathfrak{A} \triangledown \mathfrak{B}, \neg\mathfrak{A}}{\mathfrak{B}} \qquad \dfrac{\mathfrak{A} \triangledown \mathfrak{B}, \neg\mathfrak{B}}{\mathfrak{A}}$

neg $\quad \mathfrak{A} \Leftrightarrow \neg\neg\mathfrak{A} \qquad \begin{array}{l} \left| \begin{array}{l} \mathfrak{A} \\ \hline \perp \end{array} \right. \\ \neg\mathfrak{A} \end{array} \qquad \begin{array}{l} \left| \begin{array}{l} \neg\mathfrak{A} \\ \hline \perp \end{array} \right. \\ \mathfrak{A} \end{array}$

bot $\quad \dfrac{\mathfrak{A}, \neg\mathfrak{A}}{\perp}$

Each of these should remind you of rules from *ND* or *ND+*. In practice, we will allow generalized versions of cnj that let us move directly from $\mathfrak{A}_1, \mathfrak{A}_2 \ldots \mathfrak{A}_n$ to $\mathfrak{A}_1 \vartriangle \mathfrak{A}_2 \vartriangle \ldots \vartriangle \mathfrak{A}_n$. Similarly, we will allow applications of dsj and dem that skip officially required applications of neg. Thus, for example, instead of going from $\neg\mathfrak{A} \triangledown \mathfrak{B}$ to $\neg\mathfrak{A} \triangledown \neg\neg\mathfrak{B}$ and then by dem to $\neg(\mathfrak{A} \vartriangle \neg\mathfrak{B})$, we might move by dem directly from $\neg\mathfrak{A} \triangledown \mathfrak{B}$ to $\neg(\mathfrak{A} \vartriangle \neg\mathfrak{B})$. All this should become more clear as we proceed.

With definition ST and these rules, we can begin to reason about consequences of the definition. Suppose we want to show that an interpretation with $\mathsf{I}[A] = \mathsf{I}[B] = \mathsf{T}$ is such that $\mathsf{I}[\sim(A \rightarrow \sim B)] = \mathsf{T}$.

(H)

1.	$\mathsf{I}[A] = \mathsf{T}$	prem
2.	$\mathsf{I}[B] = \mathsf{T}$	prem
3.	$\mathsf{I}[\sim B] \neq \mathsf{T}$	2 ST($\sim$)
4.	$\mathsf{I}[A] = \mathsf{T} \vartriangle \mathsf{I}[\sim B] \neq \mathsf{T}$	1,3 cnj
5.	$\mathsf{I}[A \rightarrow \sim B] \neq \mathsf{T}$	4 ST($\rightarrow$)
6.	$\mathsf{I}[\sim(A \rightarrow \sim B)] = \mathsf{T}$	5 ST($\sim$)

We are given that $\mathsf{I}[A] = \mathsf{T}$ and $\mathsf{I}[B] = \mathsf{T}$. From the latter, by ST($\sim$), $\mathsf{I}[\sim B] \neq \mathsf{T}$; so $\mathsf{I}[A] = \mathsf{T}$ and $\mathsf{I}[\sim B] \neq \mathsf{T}$; so by ST($\rightarrow$), $\mathsf{I}[A \rightarrow \sim B] \neq \mathsf{T}$; so by ST($\sim$), $\mathsf{I}[\sim(A \rightarrow \sim B)] = \mathsf{T}$.

The reasoning on the left is a metalinguistic *derivation* in the sense that every step is either a premise, or justified by a definition or rule. You should be able to follow each step. On the right, we simply "tell the story" of the derivation — mirroring it step-for-step. This latter style is the one we want to develop. As we shall see, it gives us power to go beyond where the formalized derivations will take us. But the derivations serve a purpose. If we can do them, we can *use* them to construct reasoning of the sort we want. Each stage on one side corresponds to one on the other. So the derivations can guide us as we construct our reasoning, and constrain the moves we make. Note: First, on the right, we replace line references with language ("from the latter") meant to serve the same purpose. Second, the metalinguistic symbols, $\Rightarrow$, $\Leftrightarrow$, $\neg$, $\vartriangle$, $\triangledown$ are replaced with ordinary language on the right side. Finally, on the right, though we cite every *definition* when we use it, we do not cite the additional *rules* (in this case cnj). In general, as much as possible, you should strive to put the reader (and yourself at a later time) in a position to follow your reasoning — supposing just a basic familiarity with the definitions.

Consider now another example. Suppose we want to show that an interpretation with $\mathsf{I}[B] \neq \mathsf{T}$ is such that $\mathsf{I}[\sim(A \rightarrow \sim B)] \neq \mathsf{T}$.

(I)

1.	$\mathsf{I}[B] \neq \mathsf{T}$	prem
2.	$\mathsf{I}[\sim B] = \mathsf{T}$	1 ST($\sim$)
3.	$\mathsf{I}[A] \neq \mathsf{T} \triangledown \mathsf{I}[\sim B] = \mathsf{T}$	2 dsj
4.	$\mathsf{I}[A \rightarrow \sim B] = \mathsf{T}$	3 ST($\rightarrow$)
5.	$\mathsf{I}[\sim(A \rightarrow \sim B)] \neq \mathsf{T}$	4 ST($\sim$)

We are given that $\mathsf{I}[B] \neq \mathsf{T}$; so by ST($\sim$), $\mathsf{I}[\sim B] = \mathsf{T}$; so $\mathsf{I}[A] \neq \mathsf{T}$ or $\mathsf{I}[\sim B] = \mathsf{T}$; so by ST($\rightarrow$), $\mathsf{I}[A \rightarrow \sim B] = \mathsf{T}$; so by ST($\sim$), $\mathsf{I}[\sim(A \rightarrow \sim B)] \neq \mathsf{T}$.

Observe that, for a true conditional, on its right-hand side ST($\rightarrow$) *requires* a disjunction of the sort $\mathsf{I}[\mathcal{P}] \neq \mathsf{T} \triangledown \mathsf{I}[\mathcal{B}] = \mathsf{T}$. So we must obtain the disjunctive (3) in order to get (4). ST($\rightarrow$) takes a conjunction $\mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{B}] \neq \mathsf{T}$ just when the conditional is false. Do not get these cases confused, and think that somehow a conjunction of antecedent and consequent yields a true arrow! Here is another derivation of the same result, this time beginning with the opposite and breaking down to the parts, for an application of neg.

(J)

1.	$\mathsf{I}[\sim(A \rightarrow \sim B)] = \mathsf{T}$	assp
2.	$\mathsf{I}[A \rightarrow \sim B] \neq \mathsf{T}$	1 ST($\sim$)
3.	$\mathsf{I}[A] = \mathsf{T} \vartriangle \mathsf{I}[\sim B] \neq \mathsf{T}$	2 ST($\rightarrow$)
4.	$\mathsf{I}[\sim B] \neq \mathsf{T}$	3 cnj
5.	$\mathsf{I}[B] = \mathsf{T}$	4 ST($\sim$)
6.	$\mathsf{I}[B] \neq \mathsf{T}$	prem
7.	$\perp$	5,6 bot
8.	$\mathsf{I}[\sim(A \rightarrow \sim B)] \neq \mathsf{T}$	1-7 neg

Suppose $\mathsf{I}[\sim(A \rightarrow \sim B)] = \mathsf{T}$; then from ST($\sim$), $\mathsf{I}[A \rightarrow \sim B] \neq \mathsf{T}$; so by ST($\rightarrow$), $\mathsf{I}[A] = \mathsf{T}$ and $\mathsf{I}[\sim B] \neq \mathsf{T}$; so $\mathsf{I}[\sim B] \neq \mathsf{T}$; so by ST($\sim$), $\mathsf{I}[B] = \mathsf{T}$. But we are given that $\mathsf{I}[B] \neq \mathsf{T}$. This is impossible; reject the assumption: $\mathsf{I}[\sim(A \rightarrow \sim B)] \neq \mathsf{T}$.

This version takes a couple more lines. But it works as well, and provides a useful illustration of the (neg) rule. As usual, reasonings on the one side mirror that on the other. So we can use the formalized derivation as a guide for the reasoning on the right. Again, we leave out the special metalinguistic symbols. And again we cite all instances of definitions, but not the additional rules.

As you work the exercises that follow, to the extent that you can, it is good to have one line depend on the one before or in the immediate neighborhood, so as to minimize the need for extended references in the written versions. As you work these and other problems, you may find the sentential metalinguistic reference on p. 197 helpful.

E7.1. Suppose $\mathsf{I}[A] = \mathsf{T}$, $\mathsf{I}[B] \neq \mathsf{T}$ and $\mathsf{I}[C] = \mathsf{T}$. For each of the following, produce a formalized derivation, and then non-formalized reasoning to demonstrate either that it is or is not true on I. Hint: You may find a quick row of the truth table helpful to let you see which you want to show. Also, (e) is much easier than it looks.

a. $\sim B \rightarrow C$

*b. $\sim B \rightarrow \sim C$

c. $\sim[(A \rightarrow \sim B) \rightarrow \sim C]$

d. $\sim[A \rightarrow (B \rightarrow \sim C)]$

e. $\sim A \rightarrow [((A \rightarrow B) \rightarrow C) \rightarrow \sim(\sim C \rightarrow B)]$

7.2.2 Validity

So far, we have been able to reason about ST and truth. Let us now extend results to validity. For this, we need to augment our formalized system. Let 'S' be a metalinguistic existential quantifier — it asserts the existence of some *object*. For now, 'S' will appear only in contexts asserting the existence of *interpretations*. Thus, for example, $S\mathsf{I}(\mathsf{I}[\mathcal{P}] = \mathsf{T})$ says there is an interpretation I such that $\mathsf{I}[\mathcal{P}] = \mathsf{T}$, and $\neg S\mathsf{I}(\mathsf{I}[P] = \mathsf{T})$ says it is not the case that there is an interpretation I such that $\mathsf{I}[\mathcal{P}] = \mathsf{T}$. Given this, we can state SV as follows, again in positive and negative forms.

SV $\quad \neg S\mathsf{I}(\mathsf{I}[\mathcal{P}_1] = \mathsf{T} \vartriangle \ldots \vartriangle \mathsf{I}[\mathcal{P}_n] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}) \Leftrightarrow \mathcal{P}_1 \ldots \mathcal{P}_n \vDash_s \mathcal{Q}$

$\quad S\mathsf{I}(\mathsf{I}[\mathcal{P}_1] = \mathsf{T} \vartriangle \ldots \vartriangle \mathsf{I}\mathcal{P}_n] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}) \Leftrightarrow \mathcal{P}_1 \ldots \mathcal{P}_n \nvDash_s \mathcal{Q}$

These should look familiar. An argument is valid when it is *not* the case that there is some interpretation that makes the premises true and the conclusion not. An argument is invalid if there is some interpretation that makes the premises true and the conclusion not.

Again, we need rules to manipulate the new operator. In general, whenever a metalinguistic *term* $\mathfrak{t}$ first appears outside the scope of a metalinguistic quantifier, it is labeled *arbitrary* or *particular*. For the sentential case, terms will always be of the sort $\mathsf{I}, \mathsf{J} \ldots$, for *interpretations*, and labeled 'particular' when they first appear apart from the quantifier S. Say $\mathfrak{A}[\mathfrak{t}]$ is some metalinguistic expression in which term $\mathfrak{t}$ appears, and $\mathfrak{A}[\mathfrak{u}]$ is like $\mathfrak{A}[\mathfrak{t}]$ but with free instances of $\mathfrak{t}$ replaced by $\mathfrak{u}$. Perhaps $\mathfrak{A}[\mathfrak{t}]$ is $\mathsf{I}[A] = \mathsf{T}$ and $\mathfrak{A}[\mathfrak{u}]$ is $\mathsf{J}[A] = \mathsf{T}$. Then,

exs $\quad \dfrac{\mathfrak{A}[\mathfrak{u}]}{S\mathfrak{t}\mathfrak{A}[\mathfrak{t}]}$ $\mathfrak{u}$ arbitrary or particular $\qquad \dfrac{S\mathfrak{t}\mathfrak{A}[\mathfrak{t}]}{\mathfrak{A}[\mathfrak{u}]}$ $\mathfrak{u}$ particular and new

As an instance of the left-hand "introduction" rule, we might move from $\mathsf{J}[A] = \mathsf{T}$, for a J labeled either arbitrary or particular, to $S\mathsf{I}(\mathsf{I}[A] = \mathsf{T})$. If interpretation J is such that $\mathsf{J}[A] = \mathsf{T}$, then there is *some* interpretation I such that $\mathsf{I}[A] = \mathsf{T}$. For the other "exploitation" rule, we may move from $S\mathsf{I}(\mathsf{I}[A] = \mathsf{T})$ to the result that $\mathsf{J}[A] = \mathsf{T}$ so long as J is identified as *particular* and is new to the derivation, in the sense required for ∃E in chapter 6. In particular, it must be that the term does not so-far appear outside the scope of a metalinguistic quantifier, and does not appear free in the final result of the derivation. Given that some I is such that $\mathsf{I}[A] = \mathsf{T}$, we set up J as a particular interpretation for which it is so.[2]

In addition, it will be helpful to allow a rule which lets us make assertions by *inspection* about already given interpretations — and we will limit justifications by (ins) just to assertions about interpretations (and, later, variable assignments). Thus, for example, in the context of an interpretation I on which $\mathsf{I}[A] = \mathsf{T}$, we might allow,

[2] Observe that, insofar as it is quantified, term I may itself be new in the sense that it does not so far appear outside the scope of a quantifier. Thus we may be justified in moving from $S\mathsf{I}(\mathsf{I}[A] = \mathsf{T})$ to $\mathsf{I}[A] = \mathsf{T}$, with I particular. However, as a matter of style, we will typically switch terms upon application of the exs rule.

n. $\mathsf{I}[A] = \mathsf{T}$ ins (I particular)

as a line of one of our derivations. In this case, I is a *name* of the interpretation, and listed as particular on first use.

Now suppose we want to show that $(B \rightarrow \sim D), \sim B \nvDash_s D$. Recall that our strategy for showing that an argument is invalid is to *produce* an interpretation, and show that it makes the premises true and the conclusion not. So consider an interpretation J such that $\mathsf{J}[B] \neq \mathsf{T}$ and $\mathsf{J}[D] \neq \mathsf{T}$.

(K)

1.	$\mathsf{J}[B] \neq \mathsf{T}$	ins (J particular)
2.	$\mathsf{J}[B] \neq \mathsf{T} \triangledown \mathsf{J}[\sim D] = \mathsf{T}$	1 dsj
3.	$\mathsf{J}[B \rightarrow \sim D] = \mathsf{T}$	2 ST($\rightarrow$)
4.	$\mathsf{J}[\sim B] = \mathsf{T}$	1 ST($\sim$)
5.	$\mathsf{J}[D] \neq \mathsf{T}$	ins
6.	$\mathsf{J}[B \rightarrow \sim D] = \mathsf{T} \vartriangle \mathsf{J}[\sim B] = \mathsf{T} \vartriangle \mathsf{J}[D] \neq \mathsf{T}$	3,4,5 cnj
7.	$S\mathsf{I}(\mathsf{I}[B \rightarrow \sim D] = \mathsf{T} \vartriangle \mathsf{I}[\sim B] = \mathsf{T} \vartriangle \mathsf{I}[D] \neq \mathsf{T})$	6 exs
8.	$B \rightarrow \sim D, \sim B \nvDash_s D$	7 SV

(1) and (5) are by inspection of the interpretation J, where an individual name is always labeled "particular" when it first appears. At (6) we have a conclusion about interpretation J, and at (7) we generalize to the existential, for an application of SV at (8). Here is the corresponding informal reasoning.

> $\mathsf{J}[B] \neq \mathsf{T}$; so either $\mathsf{J}[B] \neq \mathsf{T}$ or $\mathsf{J}[\sim D] = \mathsf{T}$; so by ST($\rightarrow$), $\mathsf{J}[B \rightarrow \sim D] = \mathsf{T}$. But since $\mathsf{J}[B] \neq \mathsf{T}$, by ST($\sim$), $\mathsf{J}[\sim B] = \mathsf{T}$. And $\mathsf{J}[D] \neq \mathsf{T}$. So $\mathsf{J}[B \rightarrow \sim D] = \mathsf{T}$ and $\mathsf{J}[\sim B] = \mathsf{T}$ but $\mathsf{J}[D] \neq \mathsf{T}$. So there is an interpretation I such that $\mathsf{I}[B \rightarrow \sim D] = \mathsf{T}$ and $\mathsf{I}[\sim B] = \mathsf{T}$ but $\mathsf{I}[D] \neq \mathsf{T}$. So by SV, $(B \rightarrow \sim D), \sim B \nvDash_s D$

It should be clear that this reasoning reflects that of the derivation. The derivation thus constrains the steps we make, and guides us to our goal. We show the argument is invalid by showing that there exists an interpretation on which the premises are true and the conclusion is not.

Say we want to show that $\sim(A \rightarrow B) \vDash_s A$. To show that an argument is valid, our idea has been to assume otherwise, and show that the assumption leads to contradiction. So we might reason as follows.

(L)

1.	$\sim(A \rightarrow B) \not\models_s A$	assp
2.	$\mathcal{S}\mathsf{I}(\mathsf{I}[\sim(A \rightarrow B)] = \mathsf{T} \vartriangle \mathsf{I}[A] \neq \mathsf{T})$	1 SV
3.	$\mathsf{J}[\sim(A \rightarrow B)] = \mathsf{T} \vartriangle \mathsf{J}[A] \neq \mathsf{T}$	2 exs (J particular)
4.	$\mathsf{J}[\sim(A \rightarrow B)] = \mathsf{T}$	3 cnj
5.	$\mathsf{J}[A \rightarrow B] \neq \mathsf{T}$	4 ST($\sim$)
6.	$\mathsf{J}[A] = \mathsf{T} \vartriangle \mathsf{J}[B] \neq \mathsf{T}$	5 ST($\rightarrow$)
7.	$\mathsf{J}[A] = \mathsf{T}$	6 cnj
8.	$\mathsf{J}[A] \neq \mathsf{T}$	3 cnj
9.	$\perp$	7,8 bot
10.	$\sim(A \rightarrow B) \models_s A$	1-9 neg

Suppose $\sim(A \rightarrow B) \not\models_s A$; then by SV there is some I such that $\mathsf{I}[\sim(A \rightarrow B)] = \mathsf{T}$ and $\mathsf{I}[A] \neq \mathsf{T}$. Let J be a particular interpretation of this sort; then $\mathsf{J}[\sim(A \rightarrow B)] = \mathsf{T}$ and $\mathsf{J}[A] \neq \mathsf{T}$. From the former, by ST($\sim$), $\mathsf{J}[A \rightarrow B] \neq \mathsf{T}$; so by ST($\rightarrow$), $\mathsf{J}[A] = \mathsf{T}$ and $\mathsf{J}[B] \neq \mathsf{T}$. So both $\mathsf{J}[A] = \mathsf{T}$ and $\mathsf{J}[A] \neq \mathsf{T}$. This is impossible; reject the assumption: $\sim(A \rightarrow B) \models_s A$.

At (2) we have the result that there is some interpretation on which the premise is true and the conclusion is not. At (3), we set up to reason about a particular J for which this is so. J does not so-far appear in the derivation, and does not appear in the goal at (9). So we instantiate to it. This puts us in a position to reason by ST. The pattern is typical. Given that the assumption leads to contradiction, we are justified in rejecting the assumption, and thus conclude that the argument is valid. It is important that we show the argument is valid, without reasoning individually about every possible interpretation of the basic sentences!

Notice that we can also reason generally about *forms*. Here is a case of that sort.

T7.**??**s. $\models_s (\sim\mathcal{Q} \rightarrow \sim\mathcal{P}) \rightarrow [(\sim\mathcal{Q} \rightarrow \mathcal{P}) \rightarrow \mathcal{Q}]$

1.	$\nvDash_s (\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})$	assp
2.	$\mathcal{S}\mathsf{I}(\mathsf{I}[(\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})] \neq \mathsf{T})$	1 SV
3.	$\mathsf{J}[(\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})] \neq \mathsf{T}$	2 exs (J particular)
4.	$\mathsf{J}[\sim\mathcal{Q} \to \sim\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{J}[(\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q}] \neq \mathsf{T}$	3 ST($\to$)
5.	$\mathsf{J}[(\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q}] \neq \mathsf{T}$	4 cnj
6.	$\mathsf{J}[\sim\mathcal{Q} \to \mathcal{P}] = \mathsf{T} \vartriangle \mathsf{J}[\mathcal{Q}] \neq \mathsf{T}$	5 ST($\to$)
7.	$\mathsf{J}[\mathcal{Q}] \neq \mathsf{T}$	6 cnj
8.	$\mathsf{J}[\sim\mathcal{Q}] = \mathsf{T}$	7 SF($\sim$)
9.	$\mathsf{J}[\sim\mathcal{Q} \to \mathcal{P}] = \mathsf{T}$	6 cnj
10.	$\mathsf{J}[\sim\mathcal{Q}] \neq \mathsf{T} \triangledown \mathsf{J}[\mathcal{P}] = \mathsf{T}$	9 ST($\to$)
11.	$\mathsf{J}[\mathcal{P}] = \mathsf{T}$	8,10 dsj
12.	$\mathsf{J}[\sim\mathcal{Q} \to \sim\mathcal{P}] = \mathsf{T}$	4 cnj
13.	$\mathsf{J}[\sim\mathcal{Q}] \neq \mathsf{T} \triangledown \mathsf{J}[\sim\mathcal{P}] = \mathsf{T}$	12 ST($\to$)
14.	$\mathsf{J}[\sim\mathcal{P}] = \mathsf{T}$	8,13 dsj
15.	$\mathsf{J}[\mathcal{P}] \neq \mathsf{T}$	14 ST($\sim$)
16.	$\bot$	11,15 bot
17.	$\vDash_s (\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})$	1-16 neg

> Suppose $\nvDash_s (\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})$; then by SV there is some I such that $\mathsf{I}[(\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})] \neq \mathsf{T}$. Let J be a particular interpretation of this sort; then $\mathsf{J}[(\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})] \neq \mathsf{T}$; so by ST($\to$), $\mathsf{J}[\sim\mathcal{Q} \to \sim\mathcal{P}] = \mathsf{T}$ and $\mathsf{J}[(\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q}] \neq \mathsf{T}$; from the latter, by ST($\to$), $\mathsf{J}[\sim\mathcal{Q} \to \mathcal{P}] = \mathsf{T}$ and $\mathsf{J}[\mathcal{Q}] \neq \mathsf{T}$; from the latter of these, by ST($\sim$), $\mathsf{J}[\sim\mathcal{Q}] = \mathsf{T}$. Since $\mathsf{J}[\sim\mathcal{Q} \to \mathcal{P}] = \mathsf{T}$, by ST($\to$), $\mathsf{J}[\sim\mathcal{Q}] \neq \mathsf{T}$ or $\mathsf{J}[\mathcal{P}] = \mathsf{T}$; but $\mathsf{J}[\sim\mathcal{Q}] = \mathsf{T}$, so $\mathsf{J}[\mathcal{P}] = \mathsf{T}$. Since $\mathsf{J}[\sim\mathcal{Q} \to \sim\mathcal{P}] = \mathsf{T}$, by ST($\to$), $\mathsf{J}[\sim\mathcal{Q}] \neq \mathsf{T}$ or $\mathsf{J}[\sim\mathcal{P}] = \mathsf{T}$; but $\mathsf{J}[\sim\mathcal{Q}] = \mathsf{T}$, so $\mathsf{J}[\sim\mathcal{P}] = \mathsf{T}$; so by ST($\sim$), $\mathsf{J}[\mathcal{P}] \neq \mathsf{T}$. This is impossible; reject the assumption: $\vDash_s (\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})$.

Observe that the steps represented by (11) and (14) are not by cnj but by the dsj rule with $\mathfrak{A} \triangledown \mathfrak{B}$ and $\neg\mathfrak{A}$ for the result that $\mathfrak{B}$.[3] Observe also that contradictions are obtained at the *metalinguistic* level. Thus $\mathsf{J}[\mathcal{P}] = \mathsf{T}$ at (11) does not contradict $\mathsf{J}[\sim\mathcal{P}] = \mathsf{T}$ at (14). Of course, it is a short step to the result that $\mathsf{J}[\mathcal{P}] = \mathsf{T}$ and $\mathsf{J}[\mathcal{P}] \neq \mathsf{T}$ which do contradict. As a general point of strategy, it is much easier to manage a negated conditional than an unnegated one — for the negated conditional yields a conjunctive result, and the unnegated a disjunctive. Thus we begin above with the negated conditionals, and *use* the results to set up applications of dsj. This is typical. Similarly we can show,

T7.**??**s. $\mathcal{P}, \mathcal{P} \to \mathcal{Q} \vDash_s \mathcal{Q}$

[3] Or, rather, we have $\neg\mathfrak{A} \triangledown \mathfrak{B}$ and $\mathfrak{A}$ — and thus skip application of neg to obtain the proper $\neg\neg\mathfrak{A}$ for this application of dsj.

T7.**??**s. $\models_s \mathcal{P} \rightarrow (\mathcal{Q} \rightarrow \mathcal{P})$

T7.**??**s. $\models_s (\mathcal{O} \rightarrow (\mathcal{P} \rightarrow \mathcal{Q})) \rightarrow ((\mathcal{O} \rightarrow \mathcal{P}) \rightarrow (\mathcal{O} \rightarrow \mathcal{Q}))$

T7.**??**s - T7.**??**s should remind you of the axioms and rule for the sentential part of *AD* from chapter 3. These results (or, rather, analogues for the quantificational case) play an important role for things to come.

These derivations are structurally much simpler than ones you have seen before from *ND*. The challenge is accommodating new notation with the different mix of rules. Again, to show that an argument is invalid, produce an interpretation; then use it for a demonstration that there exists an interpretation that makes premises true and the conclusion not. To show that an argument is valid, suppose otherwise; then demonstrate that your assumption leads to contradiction. The derivations then provide the pattern for your informal reasoning.

E7.2. Produce a formalized derivation, and then informal reasoning to demonstrate each of the following. To show invalidity, you will have to *produce* an interpretation to which your argument refers.

*a. $A \rightarrow B, \sim A \not\models_s \sim B$

*b. $A \rightarrow B, \sim B \models_s \sim A$

c. $A \rightarrow B, B \rightarrow C, C \rightarrow D \models_s A \rightarrow D$

d. $A \rightarrow B, B \rightarrow \sim A \models_s \sim A$

e. $A \rightarrow B, \sim A \rightarrow \sim B \not\models_s \sim(A \rightarrow \sim B)$

f. $(\sim A \rightarrow B) \rightarrow A \models_s \sim A \rightarrow \sim B$

g. $\sim A \rightarrow \sim B, B \models_s \sim(B \rightarrow \sim A)$

h. $A \rightarrow B, \sim B \rightarrow A \not\models_s A \rightarrow \sim B$

i. $\not\models_s [(A \rightarrow B) \rightarrow (A \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow C]$

j. $\models_s (A \rightarrow B) \rightarrow [(B \rightarrow \sim C) \rightarrow (C \rightarrow \sim A)]$

E7.3. Provide demonstrations for T7.**??**s - T7.**??**s in the informal style. Hint: you may or may not find that truth tables, or formalized derivations, would be helpful as a guide.

7.2.3 Derived Rules

Finally, for this section on sentential forms, we expand the range of our results by means of some rules for $\Rightarrow$ and $\Leftrightarrow$.

cnd $\quad \dfrac{\mathfrak{A} \Rightarrow \mathfrak{B},\ \mathfrak{A}}{\mathfrak{B}} \qquad \begin{array}{l} \left|\begin{array}{l} \mathfrak{A} \\ \hline \\ \mathfrak{B} \end{array}\right. \\ \mathfrak{A} \Rightarrow \mathfrak{B} \end{array} \qquad \dfrac{\mathfrak{A} \Rightarrow \mathfrak{B},\ \mathfrak{B} \Rightarrow \mathfrak{C}}{\mathfrak{A} \Rightarrow \mathfrak{C}}$

bcnd $\quad \dfrac{\mathfrak{A} \Leftrightarrow \mathfrak{B},\ \mathfrak{A}}{\mathfrak{B}} \qquad \dfrac{\mathfrak{A} \Leftrightarrow \mathfrak{B},\ \mathfrak{B}}{\mathfrak{A}} \qquad \dfrac{\mathfrak{A} \Rightarrow \mathfrak{B},\ \mathfrak{B} \Rightarrow \mathfrak{A}}{\mathfrak{A} \Leftrightarrow \mathfrak{B}} \qquad \dfrac{\mathfrak{A} \Leftrightarrow \mathfrak{B},\ \mathfrak{B} \Leftrightarrow \mathfrak{C}}{\mathfrak{A} \Leftrightarrow \mathfrak{C}}$

We will also allow versions of bcnd which move from, say, $\mathfrak{A} \Leftrightarrow \mathfrak{B}$ and $\neg\mathfrak{A}$, to $\neg\mathfrak{B}$ (like NB from *ND+*). And we will allow generalized versions of these rules moving directly from, say, $\mathfrak{A} \Rightarrow \mathfrak{B}$, $\mathfrak{B} \Rightarrow \mathfrak{C}$, and $\mathfrak{C} \Rightarrow \mathfrak{D}$ to $\mathfrak{A} \Rightarrow \mathfrak{D}$; and similarly, from $\mathfrak{A} \Leftrightarrow \mathfrak{B}$, $\mathfrak{B} \Leftrightarrow \mathfrak{C}$, and $\mathfrak{C} \Leftrightarrow \mathfrak{D}$ to $\mathfrak{A} \Leftrightarrow \mathfrak{D}$. In this last case, the natural informal description is, $\mathfrak{A}$ iff $\mathfrak{B}$; $\mathfrak{B}$ iff $\mathfrak{C}$; $\mathfrak{C}$ iff $\mathfrak{D}$; so $\mathfrak{A}$ iff $\mathfrak{D}$. In real cases, however, repetition of terms can be awkward and get in the way of reading. In practice, then, the pattern collapses to, $\mathfrak{A}$ iff $\mathfrak{B}$; iff $\mathfrak{C}$; iff $\mathfrak{D}$; so $\mathfrak{A}$ iff $\mathfrak{D}$ — where this is understood as in the official version.

Also, when demonstrating that $\mathfrak{A} \Rightarrow \mathfrak{B}$, in many cases, it is helpful to get $\mathfrak{B}$ by neg; officially, the pattern is as on the left,

$$\begin{array}{l} \left|\begin{array}{l} \mathfrak{A} \\ \hline \left|\begin{array}{l} \neg\mathfrak{B} \\ \hline \\ \bot \end{array}\right. \\ \mathfrak{B} \end{array}\right. \\ \mathfrak{A} \Rightarrow \mathfrak{B} \end{array}$$

But the result is automatic once we derive a contradiction from $\mathfrak{A}$ and $\neg\mathfrak{B}$; so, in practice, this pattern collapses into:

$$\begin{array}{l} \left|\begin{array}{l} \mathfrak{A} \vartriangle \neg\mathfrak{B} \\ \hline \\ \bot \end{array}\right. \\ \mathfrak{A} \Rightarrow \mathfrak{B} \end{array}$$

So to demonstrate a conditional, it is enough to derive a contradiction from the antecedent and negation of the consequent. Let us also include among our definitions, (abv) for unpacking abbreviations. This is to be understood as justifying any biconditional $\mathfrak{A} \Leftrightarrow \mathfrak{A}'$ where $\mathfrak{A}'$ abbreviates $\mathfrak{A}$. Such a biconditional can be used as either an axiom or a rule.

We are now in a position to produce derived clauses for ST. In table form, we have already seen derived forms for ST from chapter 4. But we did not then have the official means to extend the definition.

ST′ $\quad (\wedge)\quad \mathsf{I}[\mathscr{P} \wedge \mathscr{Q}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathscr{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathscr{Q}] = \mathsf{T}$

$\qquad\qquad \mathsf{I}[\mathscr{P} \wedge \mathscr{Q}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathscr{P}] \neq \mathsf{T} \triangledown \mathsf{I}[\mathscr{Q}] \neq \mathsf{T}$

$(\vee)$ $\mathsf{I}[\mathcal{P} \vee \mathcal{Q}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T} \triangledown \mathsf{I}[\mathcal{Q}] = \mathsf{T}$
$\mathsf{I}[\mathcal{P} \vee \mathcal{Q}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] \neq \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}$

$(\leftrightarrow)$ $\mathsf{I}[\mathcal{P} \leftrightarrow \mathcal{Q}] = \mathsf{T} \Leftrightarrow (\mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] = \mathsf{T}) \triangledown (\mathsf{I}[\mathcal{P}] \neq \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T})$
$\mathsf{I}[\mathcal{P} \leftrightarrow \mathcal{Q}] \neq \mathsf{T} \Leftrightarrow (\mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}) \triangledown (\mathsf{I}[\mathcal{P}] \neq \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] = \mathsf{T})$

Again, you should recognize the derived clauses based on what you already know from truth tables.

First, consider the positive form for ST′(∧). We reason about the arbitrary interpretation. The demonstration begins by abv, and strings together biconditionals to reach the final result.

(M)

1.	$\mathsf{I}[\mathcal{P} \wedge \mathcal{Q}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\sim(\mathcal{P} \to \sim\mathcal{Q})] = \mathsf{T}$	abv (I arbitrary)
2.	$\mathsf{I}[\sim(\mathcal{P} \to \sim\mathcal{Q})] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P} \to \sim\mathcal{Q}] \neq \mathsf{T}$	ST(∼)
3.	$\mathsf{I}[\mathcal{P} \to \sim\mathcal{Q}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\sim\mathcal{Q}] \neq \mathsf{T}$	ST(→)
4.	$\mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\sim\mathcal{Q}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] = \mathsf{T}$	ST(∼)
5.	$\mathsf{I}[\mathcal{P} \wedge \mathcal{Q}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] = \mathsf{T}$	1,2,3,4 bcnd

This derivation puts together a string of biconditionals of the form $\mathfrak{A} \Leftrightarrow \mathfrak{B}$, $\mathfrak{B} \Leftrightarrow \mathfrak{C}$, $\mathfrak{C} \Leftrightarrow \mathfrak{D}$, $\mathfrak{D} \Leftrightarrow \mathfrak{E}$; the conclusion follows by bcnd. Notice that we use the abbreviation and first two definitions as axioms, to state the biconditonals. Technically, (4) results from an implicit $\mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\sim\mathcal{Q}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\sim\mathcal{Q}] \neq \mathsf{T}$ with ST(∼) as a replacement rule, substituting $\mathsf{I}[\mathcal{Q}] = \mathsf{T}$ for $\mathsf{I}[\sim\mathcal{Q}] \neq \mathsf{T}$ on the right-hand side. In the "collapsed" biconditional form, the result is as follows.

> By abv, $\mathsf{I}[\mathcal{P} \wedge \mathcal{Q}] = \mathsf{T}$ iff $\mathsf{I}[\sim(\mathcal{P} \to \sim\mathcal{Q})] = \mathsf{T}$; by ST(∼), iff $\mathsf{I}[\mathcal{P} \to \sim\mathcal{Q}] \neq \mathsf{T}$; by ST(→), iff $\mathsf{I}[\mathcal{P}] = \mathsf{T}$ and $\mathsf{I}[\sim\mathcal{Q}] \neq \mathsf{T}$; by ST(∼), iff $\mathsf{I}[\mathcal{P}] = \mathsf{T}$ and $\mathsf{I}[\mathcal{Q}] = \mathsf{T}$. So $\mathsf{I}[\mathcal{P} \wedge \mathcal{Q}] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{P}] = \mathsf{T}$ and $\mathsf{I}[\mathcal{Q}] = \mathsf{T}$.

In this abbreviated form, each stage implies the next from start to finish. But similarly, each stage implies the one before from finish to start. So one might think of it as demonstrating conditionals in both directions all at once for eventual application of bcnd. Because we have just shown a biconditional, it follows immediately that $\mathsf{I}[\mathcal{P} \wedge \mathcal{Q}] \neq \mathsf{T}$ just in case the right hand side fails — just in case one of $\mathsf{I}[\mathcal{P}] \neq \mathsf{T}$ or $\mathsf{I}[\mathcal{Q}] \neq \mathsf{T}$. However, we can also make the point directly.

> By abv, $\mathsf{I}[\mathcal{P} \wedge \mathcal{Q}] \neq \mathsf{T}$ iff $\mathsf{I}[\sim(\mathcal{P} \to \sim\mathcal{Q})] \neq \mathsf{T}$; by ST(∼), iff $\mathsf{I}[\mathcal{P} \to \sim\mathcal{Q}] = \mathsf{T}$; by ST(→), iff $\mathsf{I}[\mathcal{P}] \neq \mathsf{T}$ or $\mathsf{I}[\sim\mathcal{Q}] = \mathsf{T}$; by ST(∼), iff $\mathsf{I}[\mathcal{P}] \neq \mathsf{T}$ or $\mathsf{I}[\mathcal{Q}] \neq \mathsf{T}$. So $\mathsf{I}[\mathcal{P} \wedge \mathcal{Q}] \neq \mathsf{T}$ iff $\mathsf{I}[\mathcal{P}] \neq \mathsf{T}$ or $\mathsf{I}[\mathcal{Q}] \neq \mathsf{T}$.

Reasoning for ST′(∨) is similar. For ST′(↔) it will be helpful to introduce, as a derived rule, a sort of distribution principle.

dst $[(\neg\mathfrak{A} \triangledown \mathfrak{B}) \vartriangle (\neg\mathfrak{B} \triangledown \mathfrak{A})] \Leftrightarrow [(\mathfrak{A} \vartriangle \mathfrak{B}) \triangledown (\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})]$

To show this, our basic idea will be to obtain the conditional going in both directions, and then apply bcnd. Here is the argument from left-to-right.

1.	$[(\neg\mathfrak{A} \triangledown \mathfrak{B}) \vartriangle (\neg\mathfrak{B} \triangledown \mathfrak{A})] \vartriangle \neg[(\mathfrak{A} \vartriangle \mathfrak{B}) \triangledown (\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})]$	assp
2.	$\neg[(\mathfrak{A} \vartriangle \mathfrak{B}) \triangledown (\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})]$	1 cnj
3.	$(\neg\mathfrak{A} \triangledown \mathfrak{B}) \vartriangle (\neg\mathfrak{B} \triangledown \mathfrak{A})$	1 cnj
4.	$\neg\mathfrak{A} \triangledown \mathfrak{B}$	3 cnj
5.	$\neg\mathfrak{B} \triangledown \mathfrak{A}$	3 cnj
6.	$\neg(\mathfrak{A} \vartriangle \mathfrak{B}) \vartriangle \neg(\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})$	2 dem
7.	$\neg(\mathfrak{A} \vartriangle \mathfrak{B})$	6 cnj
8.	$\neg(\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})$	6 cnj
9.	$\neg\mathfrak{A} \triangledown \neg\mathfrak{B}$	7 dem
10.	$\mathfrak{A} \triangledown \mathfrak{B}$	8 dem
11.	$\quad \mathfrak{A}$	assp
12.	$\quad \mathfrak{B}$	4,11 dsj
13.	$\quad \neg\mathfrak{B}$	9,11 dsj
14.	$\quad \bot$	10,11 bot
15.	$\neg\mathfrak{A}$	11-14 neg
16.	$\neg\mathfrak{B}$	5,15 dsj
17.	$\mathfrak{B}$	10,15 dsj
18.	$\bot$	16,17 bot
19.	$[(\neg\mathfrak{A} \triangledown \mathfrak{B}) \vartriangle (\neg\mathfrak{B} \triangledown \mathfrak{A})] \Rightarrow [(\mathfrak{A} \vartriangle \mathfrak{B}) \triangledown (\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})]$	1-18 cnd

The conditional is demonstrated in the "collapsed" form, where we assume the antecedent with the negation of the consequent, and go for a contradiction. Note the little subderivation at (11) - (14); we have accumulated disjunctions at (4), (5), (9) and (10), but do not have any of the "sides"; often the way to make headway is to assume the negation of one side; this can feed into dsj and neg (the idea is related to SC4). Demonstration of the conditional in the other direction is left as an exercise. Given dst, you should be able to demonstrate ST($\leftrightarrow$), also in the collapsed biconditional style. You will begin by observing by abv that $\mathsf{I}[\mathcal{P} \leftrightarrow \mathcal{Q}] = \mathsf{T}$ iff $\mathsf{I}[\sim((\mathcal{P} \rightarrow \mathcal{Q}) \rightarrow \sim(\mathcal{Q} \rightarrow \mathcal{P}))] = \mathsf{T}$; by neg iff .... The negative side is relatively straightforward, and does not require dst.

Having established the derived clauses for ST$'$, we can use them directly in our reasoning. Thus, for example, let us show that $B \vee (A \wedge \sim C), (C \rightarrow A) \leftrightarrow B \nvDash_s \sim(A \wedge C)$. For this, consider an interpretation J such that $\mathsf{J}[A] = \mathsf{J}[B] = \mathsf{J}[C] = \mathsf{T}$.

Metalinguistic Quick Reference (sentential)

DEFINITIONS:

ST (∼) $\mathsf{I}[\sim\mathcal{P}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] \neq \mathsf{T}$ $\qquad$ $\mathsf{I}[\sim\mathcal{P}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T}$

(→) $\mathsf{I}[\mathcal{P} \to \mathcal{Q}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] \neq \mathsf{T} \triangledown \mathsf{I}[\mathcal{Q}] = \mathsf{T}$ $\qquad$ $\mathsf{I}[\mathcal{P} \to \mathcal{Q}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}$

(|) $\mathsf{I}[\mathcal{P} \mid \mathcal{Q}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] \neq \mathsf{T} \triangledown \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}$ $\qquad$ $\mathsf{I}[\mathcal{P} \mid \mathcal{Q}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] = \mathsf{T}$

ST′ (∧) $\mathsf{I}[\mathcal{P} \wedge \mathcal{Q}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] = \mathsf{T}$

$\mathsf{I}[\mathcal{P} \wedge \mathcal{Q}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] \neq \mathsf{T} \triangledown \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}.$

(∨) $\mathsf{I}[\mathcal{P} \vee \mathcal{Q}] = \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] = \mathsf{T} \triangledown \mathsf{I}[\mathcal{Q}] = \mathsf{T}$

$\mathsf{I}[\mathcal{P} \vee \mathcal{Q}] \neq \mathsf{T} \Leftrightarrow \mathsf{I}[\mathcal{P}] \neq \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}.$

(↔) $\mathsf{I}[\mathcal{P} \leftrightarrow \mathcal{Q}] = \mathsf{T} \Leftrightarrow (\mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] = \mathsf{T}) \triangledown (\mathsf{I}[\mathcal{P}] \neq \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T})$

$\mathsf{I}[\mathcal{P} \leftrightarrow \mathcal{Q}] \neq \mathsf{T} \Leftrightarrow (\mathsf{I}[\mathcal{P}] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}) \triangledown (\mathsf{I}[\mathcal{P}] \neq \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] = \mathsf{T}).$

SV $\neg S\mathsf{I}(\mathsf{I}[\mathcal{P}_1] = \mathsf{T} \vartriangle \ldots \vartriangle \mathsf{I}[\mathcal{P}_n] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}) \Leftrightarrow \mathcal{P}_1 \ldots \mathcal{P}_n \vDash_s \mathcal{Q}$

$S\mathsf{I}(\mathsf{I}[\mathcal{P}_1] = \mathsf{T} \vartriangle \ldots \vartriangle \mathsf{I}\mathcal{P}_n] = \mathsf{T} \vartriangle \mathsf{I}[\mathcal{Q}] \neq \mathsf{T}) \Leftrightarrow \mathcal{P}_1 \ldots \mathcal{P}_n \nvDash_s \mathcal{Q}$

abv *Abbreviation* allows $\mathfrak{A} \Leftrightarrow \mathfrak{A}'$ where $\mathfrak{A}'$ abbreviates $\mathfrak{A}$.

RULES:

com $\mathfrak{A} \triangledown \mathfrak{B} \Leftrightarrow \mathfrak{B} \triangledown \mathfrak{A}$ $\qquad$ $\mathfrak{A} \vartriangle \mathfrak{B} \Leftrightarrow \mathfrak{B} \vartriangle \mathfrak{A}$

idm $\mathfrak{A} \Leftrightarrow \mathfrak{A} \triangledown \mathfrak{A}$ $\qquad$ $\mathfrak{A} \Leftrightarrow \mathfrak{A} \vartriangle \mathfrak{A}$

dem $\neg(\mathfrak{A} \vartriangle \mathfrak{B}) \Leftrightarrow \neg\mathfrak{A} \triangledown \neg\mathfrak{B}$ $\qquad$ $\neg(\mathfrak{A} \triangledown \mathfrak{B}) \Leftrightarrow \neg\mathfrak{A} \vartriangle \neg\mathfrak{B}$

cnj $\dfrac{\mathfrak{A}, \mathfrak{B}}{\mathfrak{A} \vartriangle \mathfrak{B}}$ $\qquad$ $\dfrac{\mathfrak{A} \vartriangle \mathfrak{B}}{\mathfrak{A}}$ $\qquad$ $\dfrac{\mathfrak{A} \vartriangle \mathfrak{B}}{\mathfrak{B}}$

dsj $\dfrac{\mathfrak{A}}{\mathfrak{A} \triangledown \mathfrak{B}}$ $\qquad$ $\dfrac{\mathfrak{B}}{\mathfrak{A} \triangledown \mathfrak{B}}$ $\qquad$ $\dfrac{\mathfrak{A} \triangledown \mathfrak{B}, \neg\mathfrak{A}}{\mathfrak{B}}$ $\qquad$ $\dfrac{\mathfrak{A} \triangledown \mathfrak{B}, \neg\mathfrak{B}}{\mathfrak{A}}$

neg $\mathfrak{A} \Leftrightarrow \neg\neg\mathfrak{A}$ $\qquad$ $\left|\begin{array}{l}\mathfrak{A}\\ \hline \bot\end{array}\right.$ then $\neg\mathfrak{A}$ $\qquad$ $\left|\begin{array}{l}\neg\mathfrak{A}\\ \hline \bot\end{array}\right.$ then $\mathfrak{A}$ $\qquad$ bot $\dfrac{\mathfrak{A}, \neg\mathfrak{A}}{\bot}$

exs $\dfrac{\mathfrak{A}[\mathfrak{u}]}{S\mathfrak{t}\mathfrak{A}[\mathfrak{t}]}$ $\mathfrak{u}$ arbitrary or particular $\qquad$ $\dfrac{S\mathfrak{t}\mathfrak{A}[\mathfrak{t}]}{\mathfrak{A}[\mathfrak{u}]}$ $\mathfrak{u}$ particular and new

cnd $\dfrac{\mathfrak{A} \Rightarrow \mathfrak{B}, \mathfrak{A}}{\mathfrak{B}}$ $\qquad$ $\left|\begin{array}{l}\mathfrak{A}\\ \hline \mathfrak{B}\end{array}\right.$ then $\mathfrak{A} \Rightarrow \mathfrak{B}$ $\qquad$ $\dfrac{\mathfrak{A} \Rightarrow \mathfrak{B}, \mathfrak{B} \Rightarrow \mathfrak{C}}{\mathfrak{A} \Rightarrow \mathfrak{C}}$ $\qquad$ $\left|\begin{array}{l}\mathfrak{A} \vartriangle \neg\mathfrak{B}\\ \hline \bot\end{array}\right.$ then $\mathfrak{A} \Rightarrow \mathfrak{B}$

bcnd $\dfrac{\mathfrak{A} \Leftrightarrow \mathfrak{B}, \mathfrak{A}}{\mathfrak{B}}$ $\qquad$ $\dfrac{\mathfrak{A} \Leftrightarrow \mathfrak{B}, \mathfrak{B}}{\mathfrak{A}}$ $\qquad$ $\dfrac{\mathfrak{A} \Rightarrow \mathfrak{B}, \mathfrak{B} \Rightarrow \mathfrak{A}}{\mathfrak{A} \Leftrightarrow \mathfrak{B}}$ $\qquad$ $\dfrac{\mathfrak{A} \Leftrightarrow \mathfrak{B}, \mathfrak{B} \Leftrightarrow \mathfrak{C}}{\mathfrak{A} \Leftrightarrow \mathfrak{C}}$

dst $[(\neg\mathfrak{A} \triangledown \mathfrak{B}) \vartriangle (\neg\mathfrak{B} \triangledown \mathfrak{A})] \Leftrightarrow [(\mathfrak{A} \vartriangle \mathfrak{B}) \triangledown (\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})]$

ins *Inspection* allows assertions about interpretations and variable assignments.

(N)	1.	$\mathsf{J}[A] = \mathsf{T}$	ins (J particular)
	2.	$\mathsf{J}[C] = \mathsf{T}$	ins
	3.	$\mathsf{J}[A] = \mathsf{T} \vartriangle \mathsf{J}[C] = \mathsf{T}$	1,2 cnj
	4.	$\mathsf{J}[A \wedge C] = \mathsf{T}$	3 ST$'(\wedge)$
	5.	$\mathsf{J}[\sim(A \wedge C)] \neq \mathsf{T}$	4 ST$(\sim)$
	6.	$\mathsf{J}[B] = \mathsf{T}$	ins
	7.	$\mathsf{J}[B] = \mathsf{T} \triangledown \mathsf{J}[A \wedge \sim C] = \mathsf{T}$	6 dsj
	8.	$\mathsf{J}[B \vee (A \wedge \sim C)] = \mathsf{T}$	7 ST$'(\vee)$
	9.	$\mathsf{J}[C] \neq \mathsf{T} \triangledown \mathsf{J}[A] = \mathsf{T}$	1, dsj
	10.	$\mathsf{J}[C \rightarrow A] = \mathsf{T}$	9 ST$(\rightarrow)$
	11.	$\mathsf{J}[C \rightarrow A] = \mathsf{T} \vartriangle \mathsf{J}[B] = \mathsf{T}$	10,6 cnj
	12.	$(\mathsf{J}[C \rightarrow A] = \mathsf{T} \vartriangle \mathsf{J}[B] = \mathsf{T}) \triangledown (\mathsf{J}[C \rightarrow A] \neq \mathsf{T} \vartriangle \mathsf{J}[B] \neq \mathsf{T})$	11 dsj
	13.	$\mathsf{J}[(C \rightarrow A) \leftrightarrow B] = \mathsf{T}$	12, ST$'(\leftrightarrow)$
	14.	$\mathsf{J}[B \vee (A \wedge \sim C)] = \mathsf{T} \vartriangle \mathsf{J}[(C \rightarrow A) \leftrightarrow B] = \mathsf{T} \vartriangle \mathsf{J}[\sim(A \wedge C)] \neq \mathsf{T}$	8,13,5 cnj
	15.	$S\,\mathsf{I}[\mathsf{I}[B \vee (A \wedge \sim C)] = \mathsf{T} \vartriangle \mathsf{I}[(C \rightarrow A) \leftrightarrow B] = \mathsf{T} \vartriangle \mathsf{I}[\sim(A \wedge C)] \neq \mathsf{T}]$	14 exs
	16.	$B \vee (A \wedge \sim C), (C \rightarrow A) \leftrightarrow B \not\models_s \sim(A \wedge C)$	15 SV

Since $\mathsf{J}[A] = \mathsf{T}$ and $\mathsf{J}[C] = \mathsf{T}$, by ST$'(\wedge)$, $\mathsf{J}[A \wedge C] = \mathsf{T}$; so by ST$(\sim)$, $\mathsf{J}[\sim(A \wedge C)] \neq \mathsf{T}$. Since $\mathsf{J}[B] = \mathsf{T}$, either $\mathsf{J}[B] = \mathsf{T}$ or $\mathsf{J}[A \wedge \sim C] = \mathsf{T}$; so by ST$'(\vee)$, $\mathsf{J}[B \vee (A \wedge \sim C)] = \mathsf{T}$. Since $\mathsf{J}[A] = \mathsf{T}$, either $\mathsf{J}[C] \neq \mathsf{T}$ or $\mathsf{J}[A] = \mathsf{T}$; so by ST$(\rightarrow)$, $\mathsf{J}[C \rightarrow A] = \mathsf{T}$; so both $\mathsf{J}[C \rightarrow A] = \mathsf{T}$ and $\mathsf{J}[B] = \mathsf{T}$; so either both $\mathsf{J}[C \rightarrow A] = \mathsf{T}$ and $\mathsf{J}[B] = \mathsf{T}$ or both $\mathsf{J}[C \rightarrow A] \neq \mathsf{T}$ and $\mathsf{J}[B] \neq \mathsf{T}$; so by ST$'(\leftrightarrow)$, $\mathsf{J}[(C \rightarrow A) \leftrightarrow B] = \mathsf{T}$. So $\mathsf{J}[B \vee (A \wedge \sim C)] = \mathsf{T}$ and $\mathsf{J}[(C \rightarrow A) \leftrightarrow B] = \mathsf{T}$ but $\mathsf{J}[\sim(A \wedge C)] \neq \mathsf{T}$; so there exists an interpretation I such that $\mathsf{I}[B \vee (A \wedge \sim C)] = \mathsf{T}$ and $\mathsf{I}[(C \rightarrow A) \leftrightarrow B] = \mathsf{T}$ but $\mathsf{I}[\sim(A \wedge C)] \neq \mathsf{T}$; so by SV, $B \vee (A \wedge \sim C), (C \rightarrow A) \leftrightarrow B \not\models_s \sim(A \wedge C)$.

Similarly we can show that $A \rightarrow (B \vee C), C \leftrightarrow B, \sim C \models_s \sim A$. As usual, our strategy is to assume otherwise, and go for contradiction.

(O)

1.	$A \to (B \vee C), C \leftrightarrow B, \sim C \not\models_s \sim A$	assp
2.	$S\mathsf{I}(\mathsf{I}[A \to (B \vee C)] = \mathsf{T} \triangle \mathsf{I}[C \leftrightarrow B] = \mathsf{T} \triangle \mathsf{I}[\sim C] = \mathsf{T} \triangle \mathsf{I}[\sim A] \neq \mathsf{T})$	1 SV
3.	$\mathsf{J}[A \to (B \vee C)] = \mathsf{T} \triangle \mathsf{J}[C \leftrightarrow B] = \mathsf{T} \triangle \mathsf{J}[\sim C] = \mathsf{T} \triangle \mathsf{J}[\sim A] \neq \mathsf{T}$	2 exs (J particular)
4.	$\mathsf{J}[\sim C] = \mathsf{T}$	3 cnj
5.	$\mathsf{J}[C] \neq \mathsf{T}$	4 ST($\sim$)
6.	$\mathsf{J}[C] \neq \mathsf{T} \triangledown \mathsf{J}[B] \neq \mathsf{T}$	5 dsj
7.	$\neg(\mathsf{J}[C] = \mathsf{T} \triangle \mathsf{J}[B] = \mathsf{T})$	6 dem
8.	$\mathsf{J}[C \leftrightarrow B] = \mathsf{T}$	3 cnj
9.	$(\mathsf{J}[C] = \mathsf{T} \triangle \mathsf{J}[B] = \mathsf{T}) \triangledown (\mathsf{J}[C] \neq \mathsf{T} \triangle \mathsf{J}[B] \neq \mathsf{T})$	8 ST′($\leftrightarrow$)
10.	$\mathsf{J}[C] \neq \mathsf{T} \triangle \mathsf{J}[B] \neq \mathsf{T}$	9,7 dsj
11.	$\neg(\mathsf{J}[C] = \mathsf{T} \triangledown \mathsf{J}[B] = \mathsf{T})$	10 dem
12.	$\mathsf{J}[\sim A] \neq \mathsf{T}$	3 cnj
13.	$\mathsf{J}[A] = \mathsf{T}$	12 ST($\sim$)
14.	$\mathsf{J}[A \to (B \vee C)] = \mathsf{T}$	3 cnj
15.	$\mathsf{J}[A] \neq \mathsf{T} \triangledown \mathsf{J}[B \vee C] = \mathsf{T}$	14 ST($\to$)
16.	$\mathsf{J}[B \vee C] = \mathsf{T}$	13,15 dsj
17.	$\mathsf{J}[B] = \mathsf{T} \triangledown \mathsf{J}[C] = \mathsf{T}$	16 ST′($\vee$)
18.	$\mathsf{J}[C] = \mathsf{T} \triangledown \mathsf{J}[B] = \mathsf{T}$	17 com
19.	$\perp$	11,18 bot
20.	$A \to (B \vee C), C \leftrightarrow B, \sim C \models_s \sim A$	1-19 neg

Suppose $A \to (B \vee C)$, $C \leftrightarrow B$, $\sim C \not\models_s \sim A$; then by SV there is some I such that $\mathsf{I}[A \to (B \vee C)] = \mathsf{T}$, and $\mathsf{I}[C \leftrightarrow B] = \mathsf{T}$, and $\mathsf{I}[\sim C] = \mathsf{T}$, but $\mathsf{I}[\sim A] \neq \mathsf{T}$. Let J be a particular interpretation of this sort; then $\mathsf{J}[A \to (B \vee C)] = \mathsf{T}$, and $\mathsf{J}[C \leftrightarrow B] = \mathsf{T}$, and $\mathsf{J}[\sim C] = \mathsf{T}$, but $\mathsf{J}[\sim A] \neq \mathsf{T}$. Since $\mathsf{J}[\sim C] = \mathsf{T}$, by ST($\sim$), $\mathsf{J}[C] \neq \mathsf{T}$; so either $\mathsf{J}[C] \neq \mathsf{T}$ or $\mathsf{J}[B] \neq \mathsf{T}$; so it is not the case that both $\mathsf{J}[C] = \mathsf{T}$ and $\mathsf{J}[B] = \mathsf{T}$. But $\mathsf{J}[C \leftrightarrow B] = \mathsf{T}$; so by ST′($\leftrightarrow$), both $\mathsf{J}[C] = \mathsf{T}$ and $\mathsf{J}[B] = \mathsf{T}$, or both $\mathsf{J}[C] \neq \mathsf{T}$ and $\mathsf{J}[B] \neq \mathsf{T}$; but not the former, so $\mathsf{J}[C] \neq \mathsf{T}$ and $\mathsf{J}[B] \neq \mathsf{T}$; so it is not the case that either $\mathsf{J}[C] = \mathsf{T}$ or $\mathsf{J}[B] = \mathsf{T}$. $\mathsf{J}[\sim A] \neq \mathsf{T}$; so by ST($\sim$), $\mathsf{J}[A] = \mathsf{T}$. But $\mathsf{J}[A \to (B \vee C)] = \mathsf{T}$; so by ST($\to$), $\mathsf{J}[A] \neq \mathsf{T}$ or $\mathsf{J}[B \vee C] = \mathsf{T}$; but $\mathsf{J}[A] = \mathsf{T}$; so $\mathsf{J}[B \vee C] = \mathsf{T}$; so by ST′($\vee$), $\mathsf{J}[B] = \mathsf{T}$ or $\mathsf{J}[C] = \mathsf{T}$; so either $\mathsf{J}[C] = \mathsf{T}$ or $\mathsf{J}[B] = \mathsf{T}$. But this is impossible; reject the assumption: $A \to (B \vee C)$, $C \leftrightarrow B$, $\sim C \not\models_s \sim A$.

Though the formalized derivations are useful to discipline the way we reason, in the end, you may find the written versions to be both quicker, and easier to follow. As you work the exercises, try to free yourself from the formalized derivations to work the informal versions independently — though you should continue to use the formalized versions as a check for your work.

*E7.4. Complete the demonstration of derived clauses of ST′ by completing the demonstration for dst from right-to-left, and providing non-formalized reasonings for both the positive and negative parts of ST′($\vee$) and ST′($\leftrightarrow$).

E7.5. In the non-formalized style, show the following semantic principles for $\leftrightarrow$.

*a. *Coms*: $\mathsf{I}[\mathcal{A} \leftrightarrow \mathcal{B}] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{B} \leftrightarrow \mathcal{A}] = \mathsf{T}$.

b. *Assocs*: $\mathsf{I}[\mathcal{A} \leftrightarrow (\mathcal{B} \leftrightarrow \mathcal{C})] = \mathsf{T}$ iff $\mathsf{I}[(\mathcal{A} \leftrightarrow \mathcal{B}) \leftrightarrow \mathcal{C}] = \mathsf{T}$.

c. *Subs*: If $\mathsf{I}[\mathcal{A}] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{B}] = \mathsf{T}$, then $\mathsf{I}[\mathcal{A} \leftrightarrow \mathcal{C}] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{B} \leftrightarrow \mathcal{C}] = \mathsf{T}$.

E7.6. Using ST($\mid$) as above on p. 184, produce non-formalized reasonings to show each of the following. Again, you may or may not find formalized derivations helpful — but your reasoning should be no less clean than that guided by the rules. Hint, by ST($\mid$), $\mathsf{I}[\mathcal{P} \mid \mathcal{Q}] \neq \mathsf{T}$ iff $\mathsf{I}[\mathcal{P}] = \mathsf{T}$ and $\mathsf{I}[\mathcal{Q}] = \mathsf{T}$.

a. $\mathsf{I}[\mathcal{P} \mid \mathcal{P}] = \mathsf{T}$ iff $\mathsf{I}[\sim\mathcal{P}] = \mathsf{T}$

*b. $\mathsf{I}[\mathcal{P} \mid (\mathcal{Q} \mid \mathcal{Q})] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{P} \to \mathcal{Q}] = \mathsf{T}$

c. $\mathsf{I}[(\mathcal{P} \mid \mathcal{P}) \mid (\mathcal{Q} \mid \mathcal{Q})] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{P} \vee \mathcal{Q}] = \mathsf{T}$

d. $\mathsf{I}[(\mathcal{P} \mid \mathcal{Q}) \mid (\mathcal{P} \mid \mathcal{Q})] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{P} \wedge \mathcal{Q}] = \mathsf{T}$

E7.7. Produce non-formalized reasoning to demonstrate each of the following.

a. $A \to (B \wedge C), C \leftrightarrow B, \sim C \vDash_s \sim A$

*b. $\sim(A \leftrightarrow B), \sim A, \sim B \vDash_s C \wedge \sim C$

*c. $\sim(\sim A \wedge \sim B) \nvDash_s A \wedge B$

d. $\sim\sim A \to \sim\sim B, \sim B \to \sim A \nvDash_s B \to A$

e. $A \wedge (B \to C) \nvDash_s (A \wedge C) \vee (A \wedge B)$

f. $[(C \vee D) \wedge B] \to A, D \vDash_s B \to A$

g. $\vDash_s [A \vee ((C \to \sim B) \wedge \sim A)] \vee \sim A$

h. $D \to (A \to B), \sim A \to \sim D, C \wedge D \vDash_s B$

i. $(\sim A \vee B) \to (C \wedge D), \sim(\sim A \vee B) \nvDash_s \sim(C \wedge D)$

j. $A \wedge (B \vee C), (\sim C \vee D) \wedge (D \to \sim D) \vDash_s A \wedge B$

7.3 Quantificational

Answers to Selected Exercises

Chapter One

E1.1. Say whether each of the following stories is internally consistent or inconsistent. In either case, explain why.

a. Smoking cigarettes greatly increases the risk of lung cancer, although most people who smoke cigarettes do not get lung cancer.

   Consistent. Even though the risk of cancer goes up with smoking, it may be that most people who smoke do not have cancer. Perhaps 49% of people who smoke get cancer, and 1% of people who do not smoke get cancer. Then smoking greatly increases the risk, even though most people who smoke do not get it.

c. Abortion is always morally wrong, though abortion is morally right in order to save a woman's life.

   Inconsistent. Suppose (whether you agree or not) that abortion is *always* morally wrong. Then abortion is wrong even in the case when it would save a woman's life. So the story requires that abortion is and is not wrong.

e. No rabbits are nearsighted, though some rabbits wear glasses.

   Consistent. One reason for wearing glasses is to correct nearsightedness. But glasses may be worn for other reasons. It might be that rabbits who wear glasses are farsighted, or have astigmatism, or think that glasses are stylish. Or maybe they wear sunglasses just to look cool.

g. Barack Obama was never president of the United States, although Michelle is president right now.

   Consistent. Do not get confused by the facts! In a story it may be that Barack was never president and his wife was. Thus this story does not contradict itself and is consistent.

i. The death star is a weapon more powerful than that in any galaxy, though there is, in a galaxy far, far away, a weapon more powerful than it.

   Inconsistent. If the death star is more powerful than any weapon in any galaxy, then according to this story it is and is not more powerful than the weapon in the galaxy far far away.

E1.2. For each of the following sentences, (i) say whether it is true or false in the real world and then (ii) say, if you can, whether it is true or false according to the accompanying story. In each case, explain your answers.

Exercise 1.2

c. Sentence: After overrunning Phoenix in early 2006, a herd of buffalo overran Newark, New Jersey.

   Story: A thundering herd of buffalo overran Phoenix, Arizona in early 2006. The city no longer exists.

   (i) It is *false* in the real world that any herd of buffalo overran Newark anytime after 2006. (ii) And, though the story says something about Phoenix, the story does not contain enough information to say whether the sentence regarding Newark is true or false.

e. Sentence: Jack Nicholson has swum the Atlantic.

   Story: No human being has swum the Atlantic. Jack Nicholson and Bill Clinton and you are all human beings, and at least one of you swam all the way across!

   (i) It is *false* in the real world that Jack Nicholson has swum the Atlantic. (ii) This story is inconsistent! It requires that some human both has and has not swum the Atlantic. Thus we refuse to say that it makes the sentence true or false.

g. Sentence: Your instructor is not a human being.

   Story: No beings from other planets have ever made it to this country. However, your instructor made it to this country from another planet.

   (i) Presumably, the claim that your instructor is not a human being is *false* in the real world (assuming that you are not working by independent, or computer-aided study). (ii) But this story is inconsistent! It says both that no beings from other planets have made it to this country and that some being has. Thus we refuse to say that it makes any sentence true or false.

i. Sentence: The Yugo is the most expensive car in the world.

   Story: Jaguar and Rolls Royce are expensive cars. But the Yugo is more expensive than either of them.

   (i) The Yugo is a famously cheap automobile. So the sentence is *false* in the real world. (ii) According to the story, the Yugo is more expensive than some expensive cars. But this is not enough information to say whether it is the most expensive car in the world. So there is not enough information to say whether the sentence is true or false.

E1.3. Use our invalidity test to show that each of the following arguments is not logically valid, and so not logically sound.

*For each of these problems, different stories might do the job.

a. If Joe works hard, then he will get an 'A'
Joe will get an 'A'
———
Joe works hard

   a. In any story with premises true and conclusion false,
      1. If Joe works hard, then he will get an 'A'
      2. Joe will get an 'A'
      3. Joe does not work hard
   b. Story: Joe is very smart, and if he works hard, then he will get an 'A'. Joe will get an 'A'; however, Joe cheats and gets the 'A' without working hard.
   c. This is a consistent story that makes the premises true and the conclusion false; thus, by definition, the argument is not logically valid.
   d. Since the argument is not logically valid, by definition, it is not logically sound.

E1.4. Use our validity procedure to show that each of the following is logically valid, and decide (if you can) whether it is logically sound.

*For each of these problems, particular reasonings might take different forms.

a. If Bill is president, then Hillary is first lady
Hillary is not first lady
———
Bill is not president

   a. In any story with premises true and conclusion false,
      1. If Bill is president, then Hillary is first lady
      2. Hillary is not first lady
      3. Bill is president
   b. In any such story,
      Given (1) and (3),
      4. Hillary is first lady
      Given (2) and (4),
      5. Hillary is and is not first lady

c. So no story with the premises true and conclusion false is a consistent story; so by definition, the argument is logically valid.

d. In the real world Hillary is not first lady and Bill and Hillary are married so it is true that if Bill is president, then Hillary is first lady; so all the premises are true and by definition the argument is logically sound.

E1.5. Use our procedures to say whether the following are logically valid or invalid, and sound or unsound. Hint: You may have to do some experimenting to decide whether the arguments are logically valid or invalid — and so decide which procedure applies.

c. Some dogs have red hair
Some dogs have long hair
———
Some dogs have long, red hair

a. In any story with the premise true and conclusion false,
   1. Some dogs have red hair
   2. Some dogs have long hair
   3. No dogs have long, red hair

b. Story: There are dogs with red hair, and there are dogs with long hair. However, due to a genetic defect, no dogs have long, red hair.

c. This is a consistent story that makes the premise true and the conclusion false; thus, by definition, the argument is not logically valid.

d. Since the argument is not logically valid, by definition, it is not logically sound.

E1.6. Use our procedures to say whether the following are logically valid or invalid, and sound or unsound.

d. Cheerios are square
Chex are round
———
There is no round square

a. In any story with the premises true and conclusion false,
   1. Cheerios are square
   2. Chex are round
   3. There is a round square

Exercise 1.6.d

b. In any such story, given (3),

   4. Something is round and not round

c. So no story with the premises true and conclusion false is a consistent story; so by definition, the argument is logically valid.

d. In the real world Cheerios are not square and Chex are not round, so the premises are not true; so though the argument is valid, by definition it is not logically sound.

E1.8. Which of the following are true, and which are false? In each case, explain your answers, with reference to the relevant definitions.

c. If the conclusion of an argument is true in the real world, then the argument must be logically valid.

   False. An argument is logically valid iff there is no consistent story that makes the premises true and the conclusion false. Though the conclusion is true in the real world (and so in the real story), there may be some other story that makes the premises true and the conclusion false. If this is so, then the argument is not logically valid.

e. If a premise of an argument is false in the real world, then the argument cannot be logically valid.

   False. An argument is logically valid iff there is no consistent story that makes the premises true and the conclusion false. For logical validity, there is no requirement that every story have true premises — only that ones that do, also have true conclusions. So an argument might be logically valid, and have premises that are false in many stories, including the real story.

g. If an argument is logically sound, then its conclusion is true in the real world.

   True. An argument is logically valid iff there is no consistent story that makes the premises true and the conclusion false. An argument is logically sound iff it is logically valid and its premises are true in the real world. Since the premises are true in the real world, they hold in the real story; since the argument is valid, this story cannot be one where the conclusion is false. So the conclusion of a sound argument is true in the real world.

i. If the conclusion of an argument cannot be false (is false in no consistent story), then the argument is logically valid.

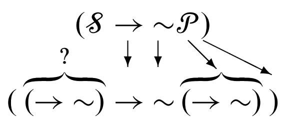
Exercise 1.8.i

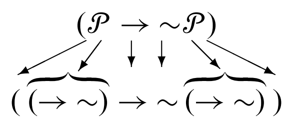
True. If there is no consistent story where the conclusion is false, then there is no consistent story where the premises are true and the conclusion is false; but an argument is logically valid iff there is no consistent story where the premises are true and the conclusion is false. So the argument is logically valid.

Chapter Two

E2.1. Assuming that $\mathscr{S}$ may represent any sentence letter, and $\mathscr{P}$ any arbitrary expression of $\mathcal{L}_s$, use maps to determine whether each of the following expressions is (i) of the form $(\mathscr{S} \rightarrow \sim\mathscr{P})$ and then (ii) whether it is of the form $(\mathscr{P} \rightarrow \sim\mathscr{P})$. In each case, explain your answers.

e. $((\rightarrow \sim) \rightarrow \sim(\rightarrow \sim))$





(i) Since $(\rightarrow \sim)$ is not a sentence letter, there is nothing to which $\mathscr{S}$ maps, and $((\rightarrow \sim) \rightarrow \sim(\rightarrow \sim))$ is not of the form $(\mathscr{S} \rightarrow \sim\mathscr{P})$. (ii) Since $\mathscr{P}$ maps to any expression, $((\rightarrow \sim) \rightarrow \sim(\rightarrow \sim))$ is of the form $(\mathscr{P} \rightarrow \sim\mathscr{P})$ by the above map.

E2.3. For each of the following expressions, demonstrate that it is a formula and a sentence of $\mathcal{L}_s$ with a tree. Then on the tree (i) bracket all the subformulas, (ii) box the immediate subformula(s), (iii) star the atomic subformulas, and (iv) circle the main operator.

a. A

subformula: [$A^\star$ This is a formula by FR(s)

In this case, the "tree" is very simple. There are no operators, and so no main operator. There are no immediate subformulas.

E2.4. Explain why the following expressions are not formulas or sentences of $\mathcal{L}_s$. Hint: you may find that an attempted tree will help you see what is wrong.

b. $(\mathscr{P} \rightarrow \mathscr{Q})$

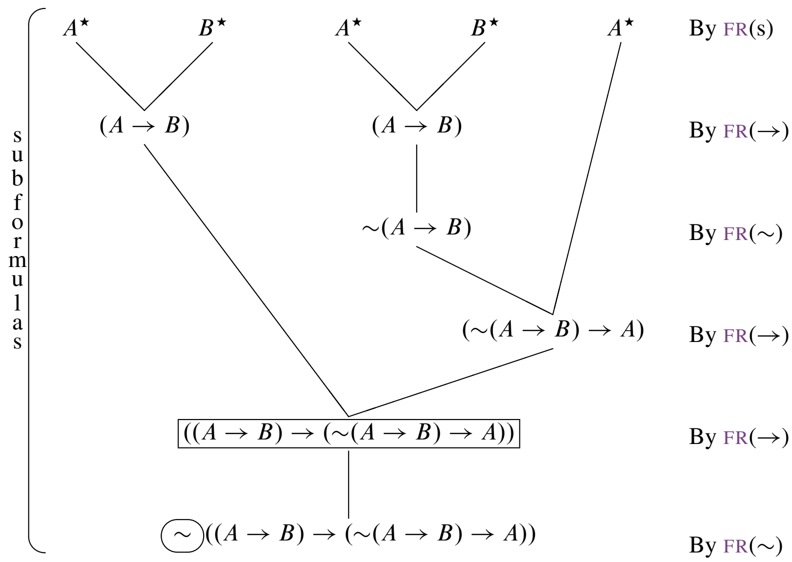
This is not a formula because $\mathscr{P}$ and $\mathscr{Q}$ are not sentence letters of $\mathcal{L}_s$. They are part of the metalanguage by which we describe $\mathcal{L}_s$, but are not among the

Roman italics (with or without subscripts) that are the sentence letters. Since it is not a formula, it is not a sentence.

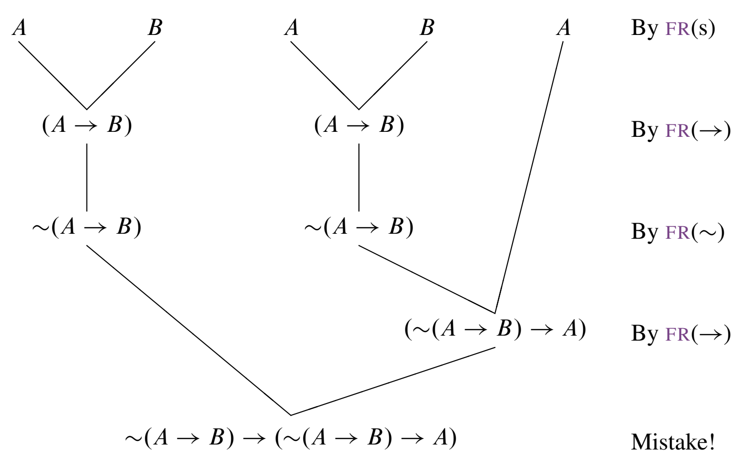
E2.5. For each of the following expressions, determine whether it is a formula and sentence of $\mathcal{L}_s$. If it is, show it on a tree, and exhibit its parts as in E2.3. If it is not, explain why as in E2.4.

a. $\sim((A \to B) \to (\sim(A \to B) \to A))$

This is a formula and a sentence.



c. $\sim(A \to B) \to (\sim(A \to B) \to A)$

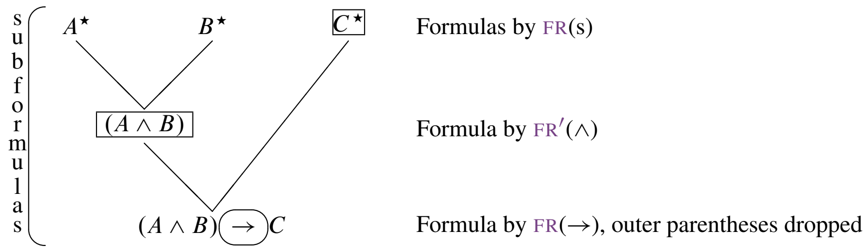


Exercise 2.5.c

Not a formula or sentence. The attempt to apply FR(→) at the last step fails, insofar as the outer parentheses are missing.

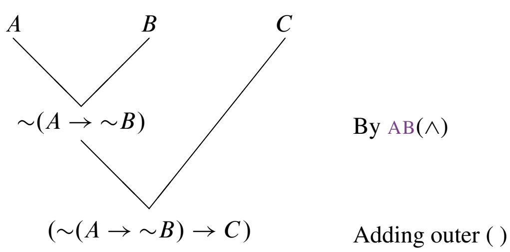
E2.6. For each of the following expressions, demonstrate that it is a formula and a sentence of $\mathcal{L}_s$ with a tree. Then on the tree (i) bracket all the subformulas, (ii) box the immediate subformula(s), (iii) star the atomic subformulas, and (iv) circle the main operator.

a. $(A \wedge B) \to C$



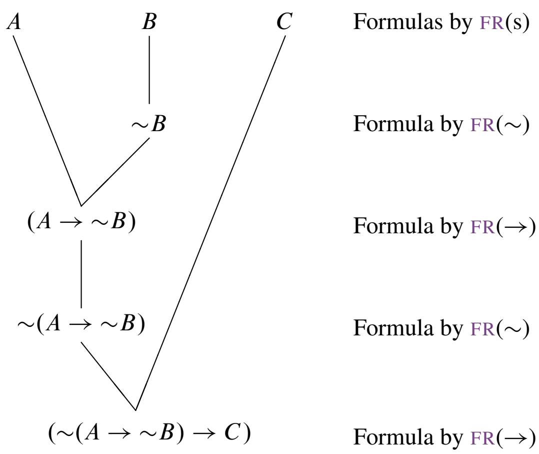
E2.7. For each of the formulas in E2.6a - e, produce an unabbreviating tree to find the unabbreviated expression it represents.

a. $(A \wedge B) \to C$



E2.8. For each of the unabbreviated expressions from E2.7a - e, produce a complete tree to show by direct application FR that it is an official formula.

a. $(\sim(A \to \sim B) \to C)$



Chapter Three

E3.1. Where *A1* is as above, construct derivations to show each of the following.

a. $\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C}) \vdash_{A1} \mathcal{B}$

1.	$\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})$	prem
2.	$[\mathcal{A} \wedge (\mathcal{B} \wedge \mathcal{C})] \to (\mathcal{B} \wedge \mathcal{C})$	$\wedge$2
3.	$\mathcal{B} \wedge \mathcal{C}$	2,1 MP
4.	$(\mathcal{B} \wedge \mathcal{C}) \to \mathcal{B}$	$\wedge$1
5.	$\mathcal{B}$	4,3 MP

E3.2. Provide derivations for T3.6, T3.7, T3.9, T3.10, T3.11, T3.12, T3.13, T3.14, T3.15, T3.16, T3.18, T3.19, T3.20, T3.21, T3.22, T3.23, T3.24, T3.25, and T3.26. As you are working these problems, you may find it helpful to refer to the *AD* summary on p. **??**.

T3.12. $\vdash_{AD} (\mathcal{A} \to \mathcal{B}) \to (\sim\sim\mathcal{A} \to \sim\sim\mathcal{B})$

1.	$\sim\sim\mathcal{A} \to \mathcal{A}$	T3.10
2.	$(\sim\sim\mathcal{A} \to \mathcal{A}) \to [(\mathcal{A} \to \mathcal{B}) \to (\sim\sim\mathcal{A} \to \mathcal{B})]$	T3.5
3.	$(\mathcal{A} \to \mathcal{B}) \to (\sim\sim\mathcal{A} \to \mathcal{B})$	2,1 MP
4.	$\mathcal{B} \to \sim\sim\mathcal{B}$	T3.11
5.	$(\mathcal{B} \to \sim\sim\mathcal{B}) \to [(\sim\sim\mathcal{A} \to \mathcal{B}) \to (\sim\sim\mathcal{A} \to \sim\sim\mathcal{B})]$	T3.4
6.	$(\sim\sim\mathcal{A} \to \mathcal{B}) \to (\sim\sim\mathcal{A} \to \sim\sim\mathcal{B})$	5,4 MP
7.	$(\mathcal{A} \to \mathcal{B}) \to (\sim\sim\mathcal{A} \to \sim\sim\mathcal{B})$	3,6 T3.2

T3.16. $\vdash_{AD} \mathcal{A} \to [\sim\mathcal{B} \to \sim(\mathcal{A} \to \mathcal{B})]$

1.	$(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{B})$	T3.1
2.	$\mathcal{A} \to [(\mathcal{A} \to \mathcal{B}) \to \mathcal{B}]$	1 T3.3
3.	$[(\mathcal{A} \to \mathcal{B}) \to \mathcal{B}] \to [\sim\mathcal{B} \to \sim(\mathcal{A} \to \mathcal{B})]$	T3.13
4.	$\mathcal{A} \to [\sim\mathcal{B} \to \sim(\mathcal{A} \to \mathcal{B})]$	2,3 T3.2

T3.21. $\mathcal{A} \to (\mathcal{B} \to \mathcal{C}) \vdash_{AD} (\mathcal{A} \wedge \mathcal{B}) \to \mathcal{C}$

1.	$\mathcal{A} \to (\mathcal{B} \to \mathcal{C})$	prem
2.	$(\mathcal{B} \to \mathcal{C}) \to (\sim\mathcal{C} \to \sim\mathcal{B})$	T3.13
3.	$\mathcal{A} \to (\sim\mathcal{C} \to \sim\mathcal{B})$	1,2 T3.2
4.	$\sim\mathcal{C} \to (\mathcal{A} \to \sim\mathcal{B})$	3, T3.3
5.	$[\sim\mathcal{C} \to (\mathcal{A} \to \sim\mathcal{B})] \to [\sim(\mathcal{A} \to \sim\mathcal{B}) \to \mathcal{C}]$	T3.14
6.	$\sim(\mathcal{A} \to \sim\mathcal{B}) \to \mathcal{C}$	5,4 MP
7.	$(\mathcal{A} \wedge \mathcal{B}) \to \mathcal{C}$	6 abv

E3.3. For each of the following, expand the derivations to include all the steps from theorems. The result should be a derivation in which each step is either a premise, an axiom, or follows from previous lines by a rule.

b. Expand the derivation for T3.4

1.	$(\mathcal{B} \to \mathcal{C}) \to [\mathcal{A} \to (\mathcal{B} \to \mathcal{C})]$	A1
2.	$[\mathcal{A} \to (\mathcal{B} \to \mathcal{C})] \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})]$	A2
3.	$([\mathcal{A} \to (\mathcal{B} \to \mathcal{C})] \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})]) \to$ $[(\mathcal{B} \to \mathcal{C}) \to ([\mathcal{A} \to (\mathcal{B} \to \mathcal{C})] \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})])]$	A1
4.	$(\mathcal{B} \to \mathcal{C}) \to ([\mathcal{A} \to (\mathcal{B} \to \mathcal{C})] \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})])$	3,2 MP
5.	$[(\mathcal{B} \to \mathcal{C}) \to ([\mathcal{A} \to (\mathcal{B} \to \mathcal{C})] \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})])] \to$ $[((\mathcal{B} \to \mathcal{C}) \to [\mathcal{A} \to (\mathcal{B} \to \mathcal{C})]) \to ((\mathcal{B} \to \mathcal{C}) \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})])]$	A2
6.	$((\mathcal{B} \to \mathcal{C}) \to [\mathcal{A} \to (\mathcal{B} \to \mathcal{C})]) \to ((\mathcal{B} \to \mathcal{C}) \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})])$	5,4 MP
7.	$(\mathcal{B} \to \mathcal{C}) \to [(\mathcal{A} \to \mathcal{B}) \to (\mathcal{A} \to \mathcal{C})]$	6,1 MP

E3.4. Consider an axiomatic system *A2* as described in the main problem. Provide derivations for each of the following, where derivations may appeal to any *prior* result (no matter what *you* have done).

a. $\mathcal{A} \to \mathcal{B}, \mathcal{B} \to \mathcal{C} \vdash_{A2} \sim(\sim\mathcal{C} \wedge \mathcal{A})$

1.	$\mathcal{A} \to \mathcal{B}$	prem
2.	$(\mathcal{A} \to \mathcal{B}) \to [\sim(\mathcal{B} \wedge \sim\mathcal{C}) \to \sim(\sim\mathcal{C} \wedge \mathcal{A})]$	A3
3.	$\sim(\mathcal{B} \wedge \sim\mathcal{C}) \to \sim(\sim\mathcal{C} \wedge \mathcal{A})$	2,1 MP
4.	$\mathcal{B} \to \mathcal{C}$	prem
5.	$\sim(\mathcal{B} \wedge \sim\mathcal{C})$	4 abv
6.	$\sim(\sim\mathcal{C} \wedge \mathcal{A})$	5,3 MP

d. $\vdash_{A2} \sim(\mathcal{A} \wedge \mathcal{B}) \rightarrow (\mathcal{B} \rightarrow \sim\mathcal{A})$

1.	$\sim\sim\mathcal{A} \rightarrow \mathcal{A}$	(c)
2.	$(\sim\sim\mathcal{A} \rightarrow \mathcal{A}) \rightarrow [\sim(\mathcal{A} \wedge \mathcal{B}) \rightarrow \sim(\mathcal{B} \wedge \sim\sim\mathcal{A})]$	A3
3.	$\sim(\mathcal{A} \wedge \mathcal{B}) \rightarrow \sim(\mathcal{B} \wedge \sim\sim\mathcal{A})$	2,1 MP
4.	$\sim(\mathcal{A} \wedge \mathcal{B}) \rightarrow (\mathcal{B} \rightarrow \sim\mathcal{A})$	3 abv

g. $\sim\mathcal{A} \rightarrow \sim\mathcal{B} \vdash_{A2} \mathcal{B} \rightarrow \mathcal{A}$

1.	$\sim\mathcal{A} \rightarrow \sim\mathcal{B}$	prem
2.	$(\sim\mathcal{A} \rightarrow \sim\mathcal{B}) \rightarrow [\sim(\sim\mathcal{B} \wedge \mathcal{B}) \rightarrow \sim(\mathcal{B} \wedge \sim\mathcal{A})]$	A3
3.	$\sim(\sim\mathcal{B} \wedge \mathcal{B}) \rightarrow \sim(\mathcal{B} \wedge \sim\mathcal{A})$	2,1 MP
4.	$\sim(\sim\mathcal{B} \wedge \mathcal{B})$	(b)
5.	$\sim(\mathcal{B} \wedge \sim\mathcal{A})$	3,4 MP
6.	$\mathcal{B} \rightarrow \mathcal{A}$	5 abv

i. $\mathcal{A} \rightarrow \mathcal{B}, \mathcal{B} \rightarrow \mathcal{C}, \mathcal{C} \rightarrow \mathcal{D} \vdash_{A2} \mathcal{A} \rightarrow \mathcal{D}$

1.	$\mathcal{A} \rightarrow \mathcal{B}$	prem
2.	$\mathcal{B} \rightarrow \mathcal{C}$	prem
3.	$\sim(\sim\mathcal{C} \wedge \mathcal{A})$	1,2 (a)
4.	$\mathcal{C} \rightarrow \mathcal{D}$	prem
5.	$(\mathcal{C} \rightarrow \mathcal{D}) \rightarrow (\sim\mathcal{D} \rightarrow \sim\mathcal{C})$	(f)
6.	$\sim\mathcal{D} \rightarrow \sim\mathcal{C}$	5,4 MP
7.	$(\sim\mathcal{D} \rightarrow \sim\mathcal{C}) \rightarrow [\sim(\sim\mathcal{C} \wedge \mathcal{A}) \rightarrow \sim(\mathcal{A} \wedge \sim\mathcal{D})]$	A3
8.	$\sim(\sim\mathcal{C} \wedge \mathcal{A}) \rightarrow \sim(\mathcal{A} \wedge \sim\mathcal{D})$	7,6 MP
9.	$\sim(\mathcal{A} \wedge \sim\mathcal{D})$	8,3 MP
10.	$\mathcal{A} \rightarrow \mathcal{D}$	9 abv

u. $\vdash_{A2} [\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})] \rightarrow [(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}]$

1.	$[(\mathcal{A} \wedge \mathcal{B}) \wedge \sim\mathcal{C}] \rightarrow [\mathcal{A} \wedge (\mathcal{B} \wedge \sim\mathcal{C})]$	(s)
2.	$(\mathcal{B} \wedge \sim\mathcal{C}) \rightarrow \sim\sim(\mathcal{B} \wedge \sim\mathcal{C})$	(e)
3.	$[\mathcal{A} \wedge (\mathcal{B} \wedge \sim\mathcal{C})] \rightarrow [\mathcal{A} \wedge \sim\sim(\mathcal{B} \wedge \sim\mathcal{C})]$	2 (q)
4.	$[(\mathcal{A} \wedge \mathcal{B}) \wedge \sim\mathcal{C}] \rightarrow [\mathcal{A} \wedge \sim\sim(\mathcal{B} \wedge \sim\mathcal{C})]$	1,3 (l)
5.	$([(\mathcal{A} \wedge \mathcal{B}) \wedge \sim\mathcal{C}] \rightarrow [\mathcal{A} \wedge \sim\sim(\mathcal{B} \wedge \sim\mathcal{C})]) \rightarrow$ $(\sim[\mathcal{A} \wedge \sim\sim(\mathcal{B} \wedge \sim\mathcal{C})] \rightarrow \sim[(\mathcal{A} \wedge \mathcal{B}) \wedge \sim\mathcal{C}])$	(f)
6.	$\sim[\mathcal{A} \wedge \sim\sim(\mathcal{B} \wedge \sim\mathcal{C})] \rightarrow \sim[(\mathcal{A} \wedge \mathcal{B}) \wedge \sim\mathcal{C}]$	5,4 MP
7.	$[\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})] \rightarrow [(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}]$	6 abv

w. $\mathcal{A} \rightarrow \mathcal{B}, \mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C}) \vdash_{A2} \mathcal{A} \rightarrow \mathcal{C}$

1.	$\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})$	prem
2.	$[\mathcal{A} \rightarrow (\mathcal{B} \rightarrow \mathcal{C})] \rightarrow [(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}]$	(u)
3.	$(\mathcal{A} \wedge \mathcal{B}) \rightarrow \mathcal{C}$	2,1 MP
4.	$\mathcal{A} \rightarrow \mathcal{A}$	(j)
5.	$\mathcal{A} \rightarrow \mathcal{B}$	prem
6.	$\mathcal{A} \rightarrow (\mathcal{A} \wedge \mathcal{B})$	4,5 (r)
7.	$\mathcal{A} \rightarrow \mathcal{C}$	6,3 (l)

Chapter Four

E4.1. Where the interpretation is as in J from p. 64, use trees to decide whether the following sentences of $\mathcal{L}_s$ are T or F.

a. $\sim A$ *false*

$A^{(\mathsf{T})}$ From J

$\sim A^{(\mathsf{F})}$ By T($\sim$), row 1

e. $\sim(A \rightarrow A)$ *false*

$A^{(\mathsf{T})}$ $A^{(\mathsf{T})}$ From J

$(A \rightarrow A)^{(\mathsf{T})}$ By T($\rightarrow$), row 1

$\sim(A \rightarrow A)^{(\mathsf{F})}$ By T($\sim$), row 1

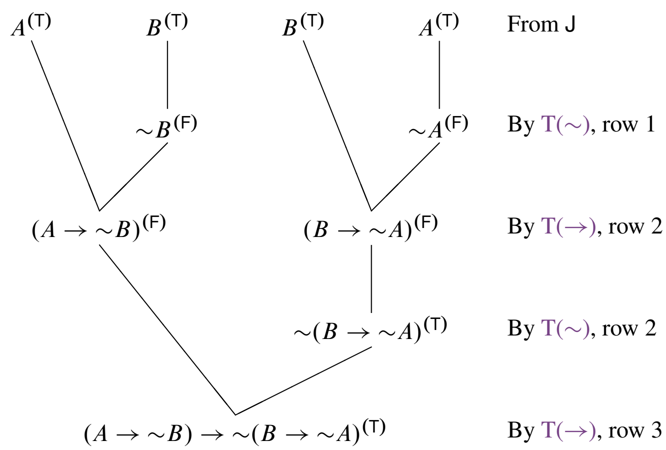
f. $(\sim A \rightarrow A)$ *true*

$A^{(\mathsf{T})}$ $A^{(\mathsf{T})}$ From J

$\sim A^{(\mathsf{F})}$ By T($\sim$), row 1

$(\sim A \rightarrow A)^{(\mathsf{T})}$ By T($\rightarrow$), row 3

i. $(A \to \sim B) \to \sim(B \to \sim A)$ *true*



E4.2. For each of the following sentences of $\mathcal{L}_s$ construct a truth table to determine its truth value for each of the possible interpretations of its basic sentences.

a. $\sim\sim A$

A	$\sim$	$\sim$	A
T	***T***	F	
F	***F***	T	

d. $(\sim B \to A) \to B$

A	B	$(\sim$	B	$\to$	$A)$	$\to$	B
T	T	F		T		***T***	
T	F	T		T		***F***	
F	T	F		T		***T***	
F	F	T		F		***T***	

g. $C \to (A \to B)$

A	B	C	C	$\to$	$(A$	$\to$	$B)$
T	T	T		***T***		T	
T	T	F		***T***		T	
T	F	T		***F***		F	
T	F	F		***T***		F	
F	T	T		***T***		T	
F	T	F		***T***		T	
F	F	T		***T***		T	
F	F	F		***T***		T	

i. $(\sim A \to B) \to (\sim C \to D)$

A	B	C	D	$(\sim$	A	$\to$	$B)$	$\to$	$(\sim$	C	$\to$	$D)$
T	T	T	T	F		T		***T***	F		T	
T	T	T	F	F		T		***T***	F		T	
T	T	F	T	F		T		***T***	T		T	
T	T	F	F	F		T		***F***	T		F	
T	F	T	T	F		T		***T***	F		T	
T	F	T	F	F		T		***T***	F		T	
T	F	F	T	F		T		***T***	T		T	
T	F	F	F	F		T		***F***	T		F	
F	T	T	T	T		T		***T***	F		T	
F	T	T	F	T		T		***T***	F		T	
F	T	F	T	T		T		***T***	T		T	
F	T	F	F	T		T		***F***	T		F	
F	F	T	T	T		F		***T***	F		T	
F	F	T	F	T		F		***T***	F		T	
F	F	F	T	T		F		***T***	T		T	
F	F	F	F	T		F		***T***	T		F	

E4.3. For each of the following, use truth tables to decide whether the entialment claims hold.

a. $A \to \sim A \vDash_s \sim A$ *valid*

A	A	$\to$	$\sim$	A	/	$\sim$	A
T	T	***F***	F	T		***F***	T
F	F	***T***	T	F		***T***	F

c. $A \to B, \sim A \vDash_s \sim B$ *invalid*

A	B	$A \to B$	$\sim A$	/	$\sim B$
T	T	***T***	***F***		***F***
T	F	***F***	***F***		***T***
F	T	***T***	***T***		***F*** ⇐
F	F	***T***	***T***		***T***

g. $\vDash_s [A \to (C \to B)] \to [(A \to C) \to (A \to B)]$ *valid*

A	B	C	$[A$	$\to$	$(C$	$\to$	$B)]$	$\to$	$[(A$	$\to$	$C)$	$\to$	$(A$	$\to$	$B)]$
T	T	T		T		T		***T***		T		T		T	
T	T	F		T		T		***T***		F		T		T	
T	F	T		F		F		***T***		T		F		F	
T	F	F		T		T		***T***		F		T		F	
F	T	T		T		T		***T***		T		T		T	
F	T	F		T		T		***T***		T		T		T	
F	F	T		T		F		***T***		T		T		T	
F	F	F		T		T		***T***		T		T		T	

E4.4. For each of the following, use truth tables to decide whether the entialment claims hold.

c. $B \vee \sim C \vDash_s B \rightarrow C$ *invalid*

B	C	$B \vee \sim C$		/	$B \rightarrow C$	
T	T	***T***	F		***T***	
T	F	***T***	T		***F***	⇐
F	T	***F***	F		***T***	
F	F	***T***	T		***T***	

d. $A \vee B, \sim C \rightarrow \sim A, \sim(B \wedge \sim C) \vDash_s C$ *valid*

A	B	C	$A \vee B$	$\sim C$	$\rightarrow$	$\sim A$	$\sim(B$	$\wedge$	$\sim C)$	/ C
T	T	T	***T***	F	***T***	F	***T***	F	F	***T***
T	T	F	***T***	T	***F***	F	***F***	T	T	***F***
T	F	T	***T***	F	***T***	F	***T***	F	F	***T***
T	F	F	***T***	T	***F***	F	***T***	F	T	***F***
F	T	T	***T***	F	***T***	T	***T***	F	F	***T***
F	T	F	***T***	T	***T***	T	***F***	T	T	***F***
F	F	T	***F***	F	***T***	T	***T***	F	F	***T***
F	F	F	***F***	T	***T***	T	***T***	F	T	***F***

h. $\vDash_s \sim(A \leftrightarrow B) \leftrightarrow (A \wedge \sim B)$ *invalid*

A	B	$\sim(A$	$\leftrightarrow B)$	$\leftrightarrow$	$(A \wedge$	$\sim B)$	
T	T	F	T	***T***	F	F	
T	F	T	F	***T***	T	T	
F	T	T	F	***F***	F	F	⇐
F	F	F	T	***T***	F	T	

E4.5. For each of the following, use truth tables to decide whether the entailment claims hold.

a. $\exists x Ax \rightarrow \exists x Bx, \sim\exists x Ax \vDash_s \exists x Bx$ *invalid*

$\exists x Ax$	$\exists x Bx$	$\exists x Ax \rightarrow \exists x Bx$	$\sim\exists x Ax$	/ $\exists x Bx$	
T	T	***T***	***F***	***T***	
T	F	***F***	***F***	***F***	
F	T	***T***	***T***	***T***	
F	F	***T***	***T***	***F***	⇐

Chapter Five

E5.1. For each of the following, identify the simple sentences that are parts. If the sentence is compound, use underlines to exhibit its operator structure, and say what is its main operator.

Exercise 5.1

h. Hermoine believes that studying is good, and Hermione studies hard, but Ron believes studying is good, and it is not the case that Ron studies hard.

Simple sentences:
Stydying is good
Hermione studies hard
Ron studies hard

Hermoine believes that studying is good and Hermione studies hard but Ron believes studying is good and it is not the case that Ron studies hard.

main operator: ____ but ____

E5.2. Which of the following operators are truth-functional and which are not? If the operator is truth-functional, display the relevant table; if it is not, give a case to show that it is not. Clearly explain your response.

a. It is a fact that ____ *truth functional*

It is a fact that	____
T	T
F	F

In any situation, the compound takes the same value as the sentence in the blank. So the operator is truth-functional.

c. ____ but ____ *truth functional*

____	but	____
T	***T***	T
T	***F***	F
F	***F***	T
F	***F***	F

In any situation this operator takes the same value as ____ and ____. Though 'but' may carry a conversational sense of opposition not present with 'and' the *truth value* of the compound works the same. Thus, where Bob loves Sue even 'Bob loves Sue but Bob loves Sue' might elicit the response "*True*, but why did you say that?"

f. It is always the case that ____ *not truth functional*

It may be that any false sentence in the blank results in a false compound. However, consider something true in the blank: perhaps 'I am at my desk' and 'Life is hard' are both true. But

It is always the case that I am at my desk

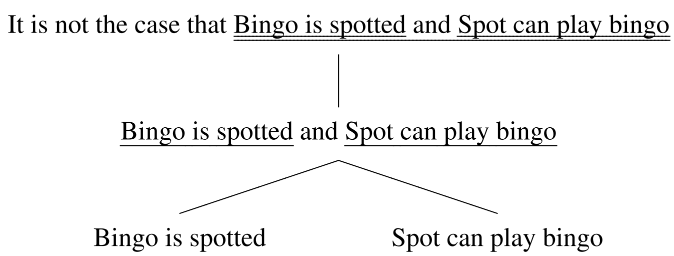
Exercise 5.2.f

It is always the case that life is hard

are such that the first is false, but the second remains true. For perhaps I sometimes get up from my desk (so that the first is false), but the difficult character of living goes on and on (and on). Thus there are situations where truth values of sentences in the blanks are the same, but the truth values of resultant compounds are different. So the operator is not truth-functional.

E5.3. Use our method to expose truth functional structure and produce parse trees for each of the following. Use your trees to produce an interpretation function for the sentences.

d. It is not the case that: Bingo is spotted and Spot can play bingo.



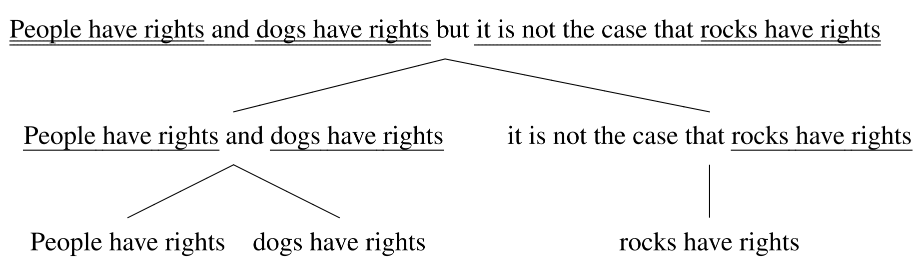
From this sentence, II includes,

B: Bingo is spotted

S: Spot can play bingo

E5.4. Use our method to expose truth functional structure and produce parse trees for each of the following. Use your trees to produce an interpretation function for the sentences.

a. People have rights and dogs have rights, but rocks do not.

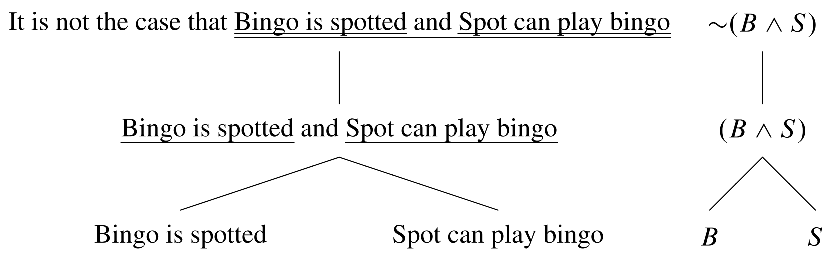


From this sentence, II includes,

P: People have rights
D: Dogs have rights
R: Rocks have rights

E5.5. Construct parallel trees to complete the translation of the sentences from E5.3 and E5.4.

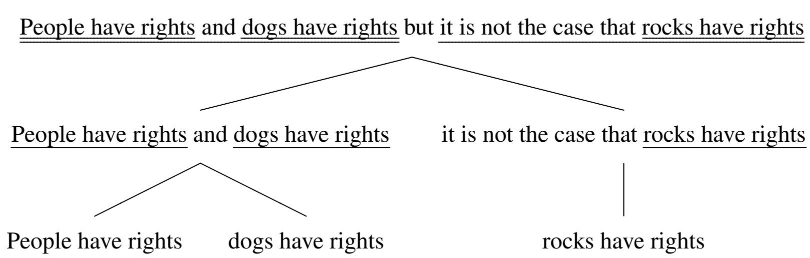
d. It is not the case that: Bingo is spotted and Spot can play bingo.



Where II includes,

B: Bingo is spotted
S: Spot can play bingo

a. People have rights and dogs have rights, but rocks do not.

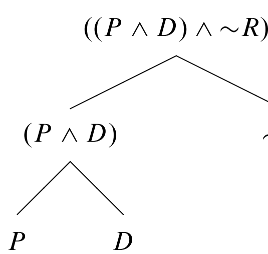


Where II includes,

P: People have rights

D: Dogs have rights

R: Rocks have rights



E5.6. Use our method to translate each of the following. That is, generate parse trees with an interpretation function for all the sentences, and then parallel trees to produce formal equivalents.

Exercise 5.6

c. It is not the case that: everything Plato, and Aristotle, and Ayn Rand said was true.

It is not the case that everything Plato said was true and everything Aristotle said was true and everything Ayn Rand said was true

everything Plato said was true and everything Aristotle said was true and everything Ayn Rand said was true

everything Plato said was true | everything Aristotle said was true and everything Ayn Rand said was true

everything Aristotle said was true | everything Ayn Rand said was true

$\sim(P \wedge (A \wedge R))$

$P \wedge (A \wedge R))$

P | $(A \wedge R)$

A | R

Where ‖ includes,

P: Everything Plato said was true

A: Everything Aristotle said was true

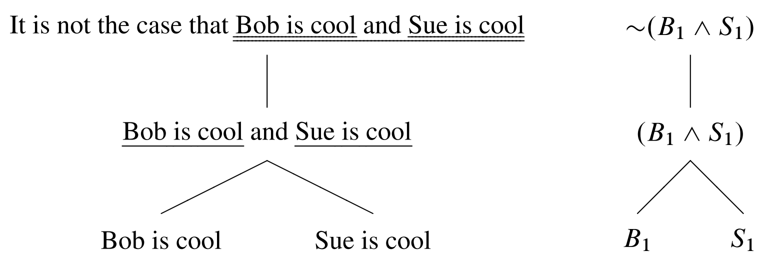
R: Everything Ayn Rand said was true

Another natural result is
$\sim((P \wedge A) \wedge R)$

Exercise 5.6.c

E5.8. Using the given interpretation function, produce parse trees and then parallel ones to complete the translation for each of the following.

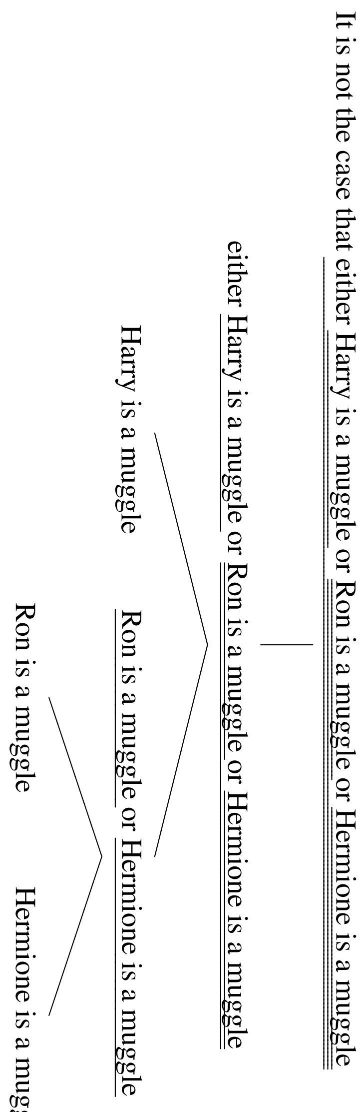
h. Not both Bob and Sue are cool.

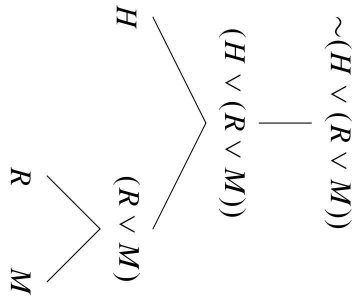


E5.9. Use our method to translate each of the following. That is, generate parse trees with an interpretation function for all the sentences, and then parallel trees to produce formal equivalents.

Exercise 5.9

d. Neither Harry, nor Ron, nor Hermione are Muggles.





Include in the interpretation function,

H: Harry is a muggle

R: Ron is a muggle

M: Hermione is a muggle

Other natural options are,
$\sim((H \vee R) \vee M)$
$(\sim H \wedge (\sim R \wedge \sim M))$
$((\sim H \wedge \sim R) \wedge \sim M)$

g. Although blatching and blagging are illegal in Quidditch, the woolongong shimmy is not.

Although It is not the case that blatching is legal in Quidditch and it is not the case that blagging is legal in Quidditch, it is not the case that it is not the case that the woolongon shimmy is legal in Quidditch

It is not the case that blatching is legal in Quidditch and it is not the case that blagging is legal in Quidditch | it is not the case that it is not the case that the woolongon shimmy is legal in Quidditch

It is not the case that blatching is legal in Quidditch | It is not the case that blagging is legal in Quidditch | It is not the case that the woolongon shimmy is legal in Quidditch

blatching is legal in Quidditch | blagging is legal in Quidditch | the woolongon shimmy is legal in Quidditch

$(\sim T \wedge \sim G) \wedge \sim\sim W)$

$(\sim T \wedge \sim G)$ | $\sim\sim W$

$\sim T$ | $\sim G$ | $\sim W$

T | G | W

Include in the interpretation function,

T: Blatching is legal in Quidditch

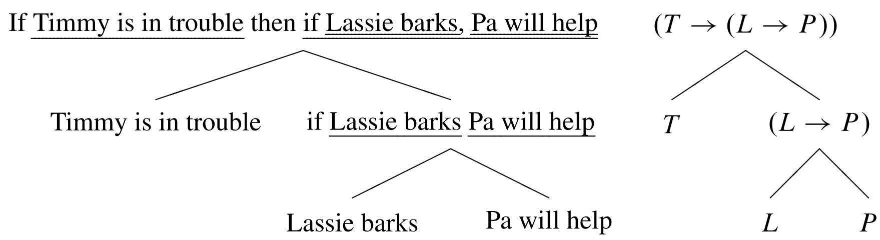
G: Blagging is legal in Quidditch

W: The woolongon shimmy is legal in Quidditch

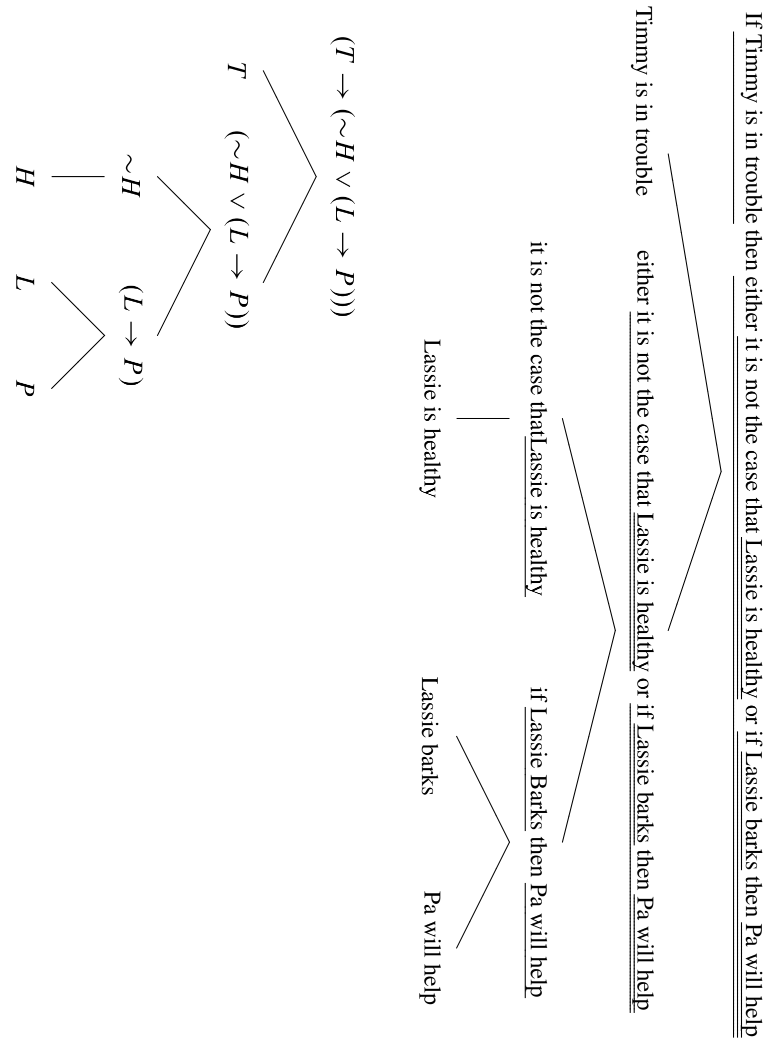
It is tempting to build 'Blatching is illegal' and so forth into the interpretation function. But this is to leave out sentential structure. And it is tempting to leave out the double negation on the right conjunct — the result is in fact equivalent, though the above translation reflects the actual claim.

E5.10. Using the given interpretation function, produce parse trees and then parallel ones to complete the translation for each of the following.

e. If Timmy is in trouble, then if Lassie barks Pa will help.



i. If Timmy is in trouble, then either Lassie is not healthy or if Lassie barks then Pa will help.



E5.11. Use our method, with or without parse trees, to produce a translation, including interpretation function for the following.

g. If you think animals do not feel pain, then vegetarianism is not right.

Include in the interpretation function,

V: Vegetarianism is right

Exercise 5.11.g

N: You think it is not the case that animals feel pain

$(N \rightarrow \sim V)$

i. Vegetarianism is right only if both animals feel pain, and animals have intrinsic value just in case they feel pain; but it is not the case that animals have intrinsic value just in case they feel pain.

Include in the interpretation function,

V: Vegetarianism is right

P: Animals feel pain

I: Animals have intrinsic value

$[V \rightarrow (P \wedge (I \leftrightarrow P))] \wedge (\sim I \leftrightarrow P)$

E5.12. For each of the following arguments: (i) Produce an adequate translation, including interpretation function and translations for the premises and conclusion. Then (ii) use truth tables to determine whether the argument is sententially valid.

a. Our car will not run unless it has gasoline
Our car has gasoline

Our car will run

Include in the interpretation function:

R: Our car will run

G: Our car has gasoline

Formal sentences:

$\sim R \vee G$
G

R

Truth table:

G	R	$\sim R$	$\vee$ G	G	/ R	
T	T	F	***T***	***T***	***T***	
T	F	T	***T***	***T***	***F***	⇐
F	T	F	***F***	***F***	***T***	
F	F	T	***T***	***F***	***F***	

Not sententially valid

Exercise 5.12.a

Chapter Six

E6.1. Show that each of the following is valid in N1. Complete (a) - (d) using just rules R1, R3 and R4. You will need an application of R2 for (e).

a. $(A \wedge B) \wedge C \vdash_{N1} A$

1.	$(A \wedge B) \wedge C$	P
2.	$A \wedge B$	1 R3
3.	A	2 R3 Win!

E6.2. (i) For each of the arguments in E6.1, use a truth table to decide if the argument is sententially valid.

a. $(A \wedge B) \wedge C \vdash_{N1} A$

A	B	C	$(A \wedge B)$	$\wedge C$	/ A
T	T	T	T	***T***	T
T	T	F	T	***F***	T
T	F	T	F	***F***	T
T	F	F	F	***F***	T
F	T	T	F	***F***	F
F	T	F	F	***F***	F
F	F	T	F	***F***	F
F	F	F	F	***F***	F

There is no row where the premise is true and the conclusion is false; so this argument is *sententially valid*.

E6.3. Consider a derivation with structure as in the main problem. For each of the lines (3), (6), (7) and (8) which lines are accessible? which subderivatrions (if any) are accessible?

	accessible lines	accessible subderivations
line 6	(1), (4), (5)	2-3

E6.4. Suppose in a derivation with structure as in E6.3 we have obtained a formula $\mathcal{A}$ on line (3). (i) On what lines would we be allowed to conclude $\mathcal{A}$ by 3 R? Suppose there is a formula $\mathcal{B}$ on line (4). (ii) On what lines would be be allowed to conclude $\mathcal{B}$ by 4 R?

(i) There are no lines on which we could conclude $\mathcal{A}$ by 3 R.

E6.6. The following are not legitimate *ND* derivations. In each case, explain why.

Exercise 6.6

a.

1.	$(A \wedge B) \wedge (C \rightarrow B)$	P
2.	A	1 $\wedge$E

This does not apply the rule to the main operator. From (1) by $\wedge$E we can get $A \wedge B$ or $C \rightarrow B$. From the first A would follow by a *second* application of the rule.

E6.7. Provide derivations to show each of the following.

b. $A \wedge B,\ B \rightarrow C \vdash_{ND} C$

1.	$A \wedge B$	P
2.	$B \rightarrow C$	P
3.	B	1 $\wedge$E
4.	C	2,3 $\rightarrow$E

e. $A \rightarrow (A \rightarrow B) \vdash_{ND} A \rightarrow B$

1.	$A \rightarrow (A \rightarrow B)$	P
2.	$\quad A$	A (g, $\rightarrow$I)
3.	$\quad A \rightarrow B$	1,2 $\rightarrow$E
4.	$\quad B$	3,2 $\rightarrow$E
5.	$A \rightarrow B$	2-4 $\rightarrow$I

h. $A \rightarrow B,\ B \rightarrow C \vdash_{ND} (A \wedge K) \rightarrow C$

1.	$A \rightarrow B$	P
2.	$B \rightarrow C$	P
3.	$\quad A \wedge K$	A (g, $\rightarrow$I)
4.	$\quad A$	3 $\wedge$E
5.	$\quad B$	1,4 $\rightarrow$E
6.	$\quad C$	2,5 $\rightarrow$E
7.	$(A \wedge K) \rightarrow C$	3-6 $\rightarrow$I

l. $A \rightarrow B \vdash_{ND} (C \rightarrow A) \rightarrow (C \rightarrow B)$

1.	$A \rightarrow B$	P
2.	$\quad C \rightarrow A$	A (g, $\rightarrow$I)
3.	$\quad\quad C$	A (g, $\rightarrow$I)
4.	$\quad\quad A$	2,3 $\rightarrow$E
5.	$\quad\quad B$	1,4 $\rightarrow$E
6.	$\quad C \rightarrow B$	3-5 $\rightarrow$I
7.	$(C \rightarrow A) \rightarrow (C \rightarrow B)$	2-6 $\rightarrow$I

Exercise 6.7.l

E6.9. The following are not legitimate *ND* derivations. In each case, explain why.

c.

1.	W	P
2.	$\mid R$	A (c, $\sim$I)
3.	$\mid\mid \sim W$	A (c, $\sim$I)
4.	$\mid\mid \bot$	1,3 $\bot$I
5.	$\sim R$	2-4 $\sim$I

There is no contradiction against the scope line for assumption R. So we are not justified in exiting the subderivation that begins on (2). The contradiction *does* justify exiting the subderivation that begins on (3) with the conclusion W by 3-4 $\sim$E. But this would still be under the scope of assumption R, and does not get us anywhere, as we already had W at line (1)!

E6.10. Produce derivations to show each of the following.

c. $\sim A \to B, \sim B \vdash_{ND} A$

1.	$\sim A \to B$	P
2.	$\sim B$	P
3.	$\mid \sim A$	A (c, $\sim$E)
4.	$\mid B$	1,3 $\to$E
5.	$\mid \bot$	4,2 $\bot$I
6.	A	3-5 $\sim$E

g. $A \vee (A \wedge B) \vdash_{ND} A$

1.	$A \vee (A \wedge B)$	P
2.	$\mid A$	A (g, 1$\vee$E)
3.	$\mid A$	2 R
4.	$\mid A \wedge B$	A (g, 1$\vee$E)
5.	$\mid A$	4 $\wedge$E
6.	A	1,2-3,4-5 $\vee$E

l. $A \to \sim B \vdash_{ND} B \to \sim A$

1.	$A \to \sim B$	P
2.	$\quad B$	A (g, $\to$I)
3.	$\quad\quad A$	A (c, $\sim$I)
4.	$\quad\quad \sim B$	1,3 $\to$E
5.	$\quad\quad \bot$	2,4 $\bot$I
6.	$\quad \sim A$	3-5 $\sim$I
7.	$B \to \sim A$	2-6 $\to$I

E6.12. Each of the following are not legitimate *ND* derivations. In each case, explain why.

c.

1.	$A \leftrightarrow B$	P
2.	A	1 $\leftrightarrow$E

$\leftrightarrow$E takes as inputs a biconditional *and* one side or the other. We cannot get A from (1) unless we already have B.

E6.13. Produce derivations to show each of the following.

a. $(A \wedge B) \leftrightarrow A \vdash_{ND} A \to B$

1.	$(A \wedge B) \leftrightarrow A$	P
2.	$\quad A$	A (g, $\to$I)
3.	$\quad A \wedge B$	1,2 $\leftrightarrow$E
4.	$\quad B$	3 $\wedge$E
5.	$A \to B$	2-4 $\to$I

e. $A \leftrightarrow (B \wedge C), B \vdash_{ND} A \leftrightarrow C$

1.	$A \leftrightarrow (B \wedge C)$	P
2.	B	P
3.	$\quad A$	A (g, $\leftrightarrow$I)
4.	$\quad B \wedge C$	1,3 $\leftrightarrow$E
5.	$\quad C$	4 $\wedge$E
6.	$\quad C$	A (g, $\leftrightarrow$I)
7.	$\quad B \wedge C$	2,6 $\wedge$I
8.	$\quad A$	1,7 $\leftrightarrow$E
9.	$A \leftrightarrow C$	3-5,6-8 $\leftrightarrow$I

k. $\vdash_{ND} \sim\sim A \leftrightarrow A$

1.	$\sim\sim A$	A (g, $\leftrightarrow$I)
2.	$\sim A$	A (c, $\sim$E)
3.	$\sim\sim A$	1 R
4.	$\perp$	2,3 $\perp$I
5.	A	2-4 $\sim$E
6.	A	A (g $\leftrightarrow$I)
7.	$\sim A$	A (g, $\sim$I)
8.	A	6 R
9.	$\perp$	8,7 $\perp$I
10.	$\sim\sim A$	7-9 $\sim$I
11.	$\sim\sim A \leftrightarrow A$	1-5,6-10 $\leftrightarrow$I

E6.14. For each of the following, (i) which primary strategy applies? and (ii) what is the next step? If the strategy calls for a new subgoal, show the subgoal; if it calls for a subderivation, set up the subderivation. In each case, *explain* your response.

c.

1.	$\sim A \leftrightarrow B$	P
	$B \leftrightarrow \sim A$	

(i) There is no contradiction in accessible lines so SG1 does not apply. There is no disjunction in accessible lines so SG2 does not apply. The goal does not appear in the premises so SG3 does not apply. (ii) Given this, we apply SG4 and go for the goal by $\leftrightarrow$I. For this goal $\leftrightarrow$I requires a pair of subderivations which set up as follows.

1.	$\sim A \leftrightarrow B$	P
2.	B	A (g $\leftrightarrow$I)
	$\sim A$	
	$\sim A$	A (g $\leftrightarrow$I)
	B	
	$B \leftrightarrow \sim A$	_, _ $\leftrightarrow$I

E6.15. Produce derivations to show each of the following. No worked out answers are provided. However, if you get stuck, you will find strategy hints in the back.

a. $A \leftrightarrow (A \rightarrow B) \vdash_{ND} A \rightarrow B$

Hint: There is no contradiction or disjunction; and the goal is not in the premises. So set up to get the primary goal by →I in application of SG4.

b. $(A \vee B) \rightarrow (B \leftrightarrow D), B \vdash_{ND} B \wedge D$

Hint: There is no contradiction or disjunction; and the goal is not in the premises. So plan to get the primary goal by ∧I in application of SG4. Then it is a matter of SG3 to get the parts.

c. $\sim(A \wedge C), \sim(A \wedge C) \leftrightarrow B \vdash_{ND} A \vee B$

Hint: There is no contradiction or disjunction; and the goal is not in the premises. So plan to get the primary goal by (one form of) ∨I in application of SG4.

d. $A \wedge (C \wedge \sim B), (A \vee D) \rightarrow \sim E \vdash_{ND} \sim E$

Hint: There is no contradiction or disjunction; but the goal exists in the premises. So proceed by application of SG3.

e. $A \rightarrow B, B \rightarrow C \vdash_{ND} A \rightarrow C$

Hint: There is no contradiction or disjunction; and the goal is not in the premises. So set up to get the primary goal by →I in application of SG4.

f. $(A \wedge B) \rightarrow (C \wedge D) \vdash_{ND} [(A \wedge B) \rightarrow C] \wedge [(A \wedge B) \rightarrow D]$

Hint: There is no contradiction or disjunction; and the goal is not in the premises. So set up to get the primary goal by ∧I in application of SG4. Then apply SG4 and →I again for your new subgoals.

g. $A \rightarrow (B \rightarrow C), (A \wedge D) \rightarrow E, C \rightarrow D \vdash_{ND} (A \wedge B) \rightarrow E$

Hint: There is no contradiction or disjunction; and the goal is not in the premises. So set up to get the primary goal by →I in application of SG4. Then it is a matter of SG3.

h. $(A \rightarrow B) \wedge (B \rightarrow C), [(D \vee E) \vee H] \rightarrow A, \sim(D \vee E) \wedge H \vdash_{ND} C$

Hint: There is no contradiction or disjunction; but the goal is in the premises. So proceed by application of SG3.

i. $A \to (B \wedge C), \sim C \vdash_{ND} \sim(A \wedge D)$

Hint: There is no contradiction or disjunction; and the goal is not in the premises. So set up to get the primary goal by $\sim$I in application of SG4.

j. $A \to (B \to C), D \to B \vdash_{ND} A \to (D \to C)$

Hint: There is no contradiction or disjunction; and the goal is not in the premises. So set up to get the primary goal by $\to$I in application of SG4. Similar reasoning applies to the secondary goal.

k. $A \to (B \to C) \vdash_{ND} \sim C \to \sim(A \wedge B)$

Hint: There is no contradiction or disjunction; and the goal is not in the premises. So set up to get the primary goal by $\to$I in application of SG4. You can also apply SG4 to the secondary goal.

l. $(A \wedge \sim B) \to \sim A \vdash_{ND} A \to B$

Hint: There is no simple contradiction or disjunction; and the goal is not in the premises. So set up to get the primary goal by $\to$I in application of SG4. This time the secondary goal has no operator, and so falls all the way through to SG5.

m. $\sim B \leftrightarrow A, C \to B, A \wedge C \vdash_{ND} \sim K$

Hint: There is no contradiction or disjunction; and the goal is not in the premises. So set up to get the primary goal by $\sim$I in application of SG4. This works because the premises are themselves inconsistent.

n. $\sim A \vdash_{ND} A \to B$

Hint: After you set up for the main goal, look for an application of SG1.

o. $\sim A \leftrightarrow \sim B \vdash_{ND} A \leftrightarrow B$

Hint: After you set up for the main goal, look for applications of SG5.

p. $(A \vee B) \vee C, B \leftrightarrow C \vdash_{ND} C \vee A$

Hint: This is not hard, if you recoginize each of the places where SG2 applies.

q. $\vdash_{ND} A \to (A \vee B)$

Hint: Do not panic. Without premises, there is definately no contradiction or disjunction; and the goal is not in accessible lines! So set up to get the primary goal by $\to$I in application of SG4.

r. $\vdash_{ND} A \to (B \to A)$

Hint: Apply SG4 to get the goal, and again for the subgoal.

s. $\vdash_{ND} (A \leftrightarrow B) \to (A \to B)$

Hint: This requires multiple applications of SG4.

t. $\vdash_{ND} (A \wedge \sim A) \to (B \wedge \sim B)$

Hint: Once you set up for the main goal, look for an application of SG1.

u. $\vdash_{ND} (A \to B) \to [(C \to A) \to (C \to B)]$

Hint: This requires multiple applications of SG4.

v. $\vdash_{ND} [(A \to B) \wedge \sim B] \to \sim A$

Hint: Apply SG4 to get the main goal, and again to get the subgoal.

w. $\vdash_{ND} A \to [B \to (A \to B)]$

Hint: This requires multiple applications of SG4.

x. $\vdash_{ND} \sim A \to [(B \wedge A) \to C]$

Hint: After a couple applications of SG4, you will have occaision to make use of SG1 — or equivalently, SG5.

y. $\vdash_{ND} (A \to B) \to [\sim B \to \sim(A \wedge D)]$

Hint: This requires multiple applications of SG4.

E6.16. Produce derivations to demonstrate each of T6.1 - T6.20.

T6.3. $\vdash_{ND} (\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})$

1.	$\sim\mathcal{Q} \to \sim\mathcal{P}$	A (g, $\to$I)
2.	$\sim\mathcal{Q} \to \mathcal{P}$	A (g, $\to$I)
3.	$\sim\mathcal{Q}$	A (c, $\sim$E)
4.	$\mathcal{P}$	2,3 $\to$E
5.	$\sim\mathcal{P}$	1,3 $\to$E
6.	$\perp$	4,5 $\perp$I
7.	$\mathcal{Q}$	3-6 $\sim$E
8.	$(\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q}$	2-7 $\to$I
9.	$(\sim\mathcal{Q} \to \sim\mathcal{P}) \to ((\sim\mathcal{Q} \to \mathcal{P}) \to \mathcal{Q})$	1-8 $\to$I

Exercise 6.16 T6.3

T6.13. $\vdash_{ND} (\mathcal{A} \vee \mathcal{B}) \leftrightarrow (\mathcal{B} \vee \mathcal{A})$

1.	$\mathcal{A} \vee \mathcal{B}$	A (g, $\leftrightarrow$I)
2.	$\quad\mathcal{A}$	A (g, 1$\vee$E)
3.	$\quad\mathcal{B} \vee \mathcal{A}$	2 $\vee$I
4.	$\quad\mathcal{B}$	A (g, 1$\vee$E)
5.	$\quad\mathcal{B} \vee \mathcal{A}$	4 $\vee$I
6.	$\mathcal{B} \vee \mathcal{A}$	1,2-3,4-5 $\vee$E
7.	$\mathcal{B} \vee \mathcal{A}$	A (g, $\leftrightarrow$I)
8.	$\quad\mathcal{B}$	A (g, 7$\vee$E)
9.	$\quad\mathcal{A} \vee \mathcal{B}$	8 $\vee$I
10.	$\quad\mathcal{A}$	A (g, 7$\vee$E)
11.	$\quad\mathcal{A} \vee \mathcal{B}$	10 $\vee$I
12.	$\mathcal{A} \vee \mathcal{B}$	7,8-9,10-11 $\vee$E
13.	$(\mathcal{A} \vee \mathcal{B}) \leftrightarrow (\mathcal{B} \vee \mathcal{A})$	1-6,7-12 $\leftrightarrow$I

E6.17. Each of the following begins with a simple application of $\sim$I or $\sim$E for SG4 or SG5. Complete the derivations, and *explain* your use of secondary strategy.

a.

1.	$A \wedge B$	P
2.	$\sim(A \wedge C)$	P
3.	$\quad C$	A (c, $\sim$I)
	$\quad\bot$	
	$\sim C$	

1.	$A \wedge B$	P
2.	$\sim(A \wedge C)$	P
3.	$\quad C$	A (c, $\sim$I)
4.	$\quad A$	1 $\wedge$E
5.	$\quad A \wedge C$	4,3 $\wedge$I
6.	$\quad\bot$	5,2 $\bot$I
7.	$\sim C$	3-6 $\sim$I

There is no contradiction by atomics and negated atomics. And there is no disjunction in the scope of the assumption for $\sim$I. So we fall through to SC3. For this set the opposite of (2) as goal, and use primary strategies for it. The derivation of $A \wedge C$ is easy.

E6.18. Produce derivations to show each of the following. No worked out answers are provided. However, if you get stuck, you will find strategy hints in the back.

a. $A \rightarrow \sim(B \wedge C),\ B \rightarrow C \vdash_{ND} A \rightarrow \sim B$

Apply primary strategies for $\rightarrow$I and $\sim$I. Then there will be occasion for a simple application of SC3.

b. $\vdash_{ND} \sim(A \rightarrow A) \rightarrow A$

Apply primary strategies for →I and ∼E. Then there will be occasion for a simple application of SC3.

c. $A \vee B \vdash_{ND} \sim(\sim A \wedge \sim B)$

This requires no more than SC1, if you follow the primary strategies properly. From the start, apply sg2 to go for the whole goal $\sim(\sim A \wedge \sim B)$ by ∨E.

d. $\sim(A \wedge B), \sim(A \wedge \sim B) \vdash_{ND} \sim A$

You will go for the main goal by ∼I in an instance of SG4. Then it is easiest to see this as a case where you use the premises for separate instances of SC3. It is, however, also possible to see the derivation along the lines of SC4.

e. $\vdash_{ND} A \vee \sim A$

For your primary strategy, fall all the way through to SG5. Then you will be able to see the derivation either along the lines of SC3 or 4, building up to the opposite of $\sim(A \vee \sim A)$ twice.

f. $\vdash_{ND} A \vee (A \rightarrow B)$

Your primary strategy falls through to SG5. Then $\sim A$ is sufficient to prove $A \rightarrow B$, and this turns into a pure version of the pattern (AQ) for formulas with main operator ∨.

g. $A \vee \sim B, \sim A \vee \sim B \vdash_{ND} \sim B$

For this you will want to apply SG2 to one of the premises (it does not matter which) for the goal. This gives you a pair of subderivations. One is easy. In the other, SG2 applies again!

h. $A \leftrightarrow (\sim B \vee C), B \rightarrow C \vdash_{ND} A$

The goal is in the premises, so your primary strategy is SG3. The real challenge is getting $\sim B \vee C$. For this you will fall through to SG5, and assume its negation. Then the derivation can be conceived either along the lines of SC3 or SC4, and on the standard pattern for disjunctions.

i. $A \leftrightarrow B \vdash_{ND} (C \leftrightarrow A) \leftrightarrow (C \leftrightarrow B)$

Applying SG4, set up for the primary goal by ↔I. You will then need ↔I for the subgoals as well.

Exercise 6.18.i

j. $A \leftrightarrow {\sim}(B \leftrightarrow {\sim}C), {\sim}(A \vee B) \vdash_{ND} C$

Fall through to SG5 for the primary goal. Then you can think of the derivation along the lines of SC3 or SC4. The derivation of $A \vee B$ works on the standard pattern, insofar as with the assumption ${\sim}C$, ${\sim}A$ gets you B.

k. $[C \vee (A \vee B)] \wedge (C \rightarrow E), A \rightarrow D, D \rightarrow {\sim}A \vdash_{ND} C \vee B$

Though officially there is no formula with main operator $\vee$, a minor reshuffle exposes $C \vee (A \vee B)$ on an accessible line. Then the derivation is naturally driven by applications of SG2.

l. ${\sim}(A \rightarrow B), {\sim}(B \rightarrow C) \vdash_{ND} {\sim}D$

Go for the main goal by ~I in applicaiton of SG4. Then it is most natural to see the derivation as involving two separate applications of SC3. It is also possible to set the derivation up along the lines of SC4, though this leads to a rather different result.

m. $C \rightarrow {\sim}A, {\sim}(B \wedge C) \vdash_{ND} (A \vee B) \rightarrow {\sim}C$

Go for the primary goal by →I in application of SG4. Then you will need to apply SG2 to reach the subgoal.

n. ${\sim}(A \leftrightarrow B) \vdash_{ND} {\sim}A \leftrightarrow B$

Go for the primary goal by ↔I in application of SG4. You can go for one subgoal by ~E, the other by ~I. Then fall through to SC3 for the conradictions, where this will involve you in further instances of ↔I. The derivation is long, but should be straightforward if you follow the strategies.

o. $A \leftrightarrow B, B \leftrightarrow {\sim}C \vdash_{ND} {\sim}(A \leftrightarrow C)$

Go for the primary goal by ~I in application of SG4. Then the contradiction comes by application of SC4.

p. $A \vee B, {\sim}B \vee C, {\sim}C \vdash_{ND} A$

This will set up as a couple instances of ∨E. If you begin with $A \vee B$, one subderivation is easy. In the second, be on the lookout for a couple instances of SG1.

q. $({\sim}A \vee C) \vee D, D \rightarrow {\sim}B \vdash_{ND} (A \wedge B) \rightarrow C$

Officially, the primary strategy should be ∨E in application of SG2. However, in this case it will not hurth to begin with →I, and set up ∨E inside the subderivation for that.

Exercise 6.18.q

r. $A \vee D, \sim D \leftrightarrow (E \vee C), (C \wedge B) \vee [C \wedge (F \rightarrow C)] \vdash_{ND} A$

The two disjunctions require applications of SG2. In fact, there are ways to simplify this from the mechanical version entirely driven by the strategy.

s. $(A \vee B) \vee (C \wedge D), (A \leftrightarrow E) \wedge (B \rightarrow F), G \leftrightarrow \sim(E \vee F), C \rightarrow B \vdash_{ND} \sim G$

This derivation is driven by ∨E in application of SG2 and then SC3. Again, there are ways to make the derivation relatively more elegant.

t. $(A \vee B) \wedge \sim C, \sim C \rightarrow (D \wedge \sim A), B \rightarrow (A \vee E) \vdash_{ND} E \vee F$

Since there is no F in the premises, it makes sense to think the conclusion is true because E is true. So it is safe to set up to get the conclusion from E by ∨I. After some simplification, the overall strategy is revealed to be ∨E based on $A \vee B$, in application of SG2. One subderivation has another formula with main operator ∨, and so another instance of ∨E.

E6.19. Produce derivations to demonstrate each of T6.21 - T6.28.

T6.21. $\vdash_{ND} \sim(\mathcal{A} \wedge \mathcal{B}) \leftrightarrow (\sim\mathcal{A} \vee \sim\mathcal{B})$

1.	$\sim(\mathcal{A} \wedge \mathcal{B})$	A (g, $\leftrightarrow$I)
2.	$\sim(\sim\mathcal{A} \vee \sim\mathcal{B})$	A (c, $\sim$E)
3.	$\sim\mathcal{A}$	A (c, $\sim$E)
4.	$\sim\mathcal{A} \vee \sim\mathcal{B}$	3 $\vee$I
5.	$\bot$	4,2 $\bot$I
6.	$\mathcal{A}$	3-5 $\sim$E
7.	$\sim\mathcal{B}$	A (c, $\sim$E)
8.	$\sim\mathcal{A} \vee \sim\mathcal{B}$	7 $\vee$I
9.	$\bot$	8,2 $\bot$I
10.	$\mathcal{B}$	7-9 $\sim$E
11.	$\mathcal{A} \wedge \mathcal{B}$	6,10 $\wedge$I
12.	$\bot$	11,1 $\bot$I
13.	$\sim\mathcal{A} \vee \sim\mathcal{B}$	2-12 $\sim$E
14.	$\sim\mathcal{A} \vee \sim\mathcal{B}$	A (g, $\leftrightarrow$I)
15.	$\sim\mathcal{A}$	A (g, 14$\vee$E)
16.	$\mathcal{A} \wedge \mathcal{B}$	A (c, $\sim$I)
17.	$\mathcal{A}$	16 $\wedge$E
18.	$\bot$	17,15 $\bot$I
19.	$\sim(\mathcal{A} \wedge \mathcal{B})$	16-18 $\sim$I
20.	$\sim\mathcal{B}$	A (g, 14$\vee$E)
21.	$\mathcal{A} \wedge \mathcal{B}$	A (c, $\sim$I)
22.	$\mathcal{B}$	21 $\wedge$E
23.	$\bot$	22,20 $\bot$I
24.	$\sim(\mathcal{A} \wedge \mathcal{B})$	21-23 $\sim$I
25.	$\sim(\mathcal{A} \wedge \mathcal{B})$	14,15-19,20-24 $\vee$E
26.	$\sim(\mathcal{A} \wedge \mathcal{B}) \leftrightarrow (\sim\mathcal{A} \vee \sim\mathcal{B})$	1-13,14-25 $\leftrightarrow$I

Chapter Seven

E7.1. Suppose $\mathsf{I}[A] = \mathsf{T}$, $\mathsf{I}[B] \neq \mathsf{T}$ and $\mathsf{I}[C] = \mathsf{T}$. For each of the following, produce a formalized derivation, and then non-formalized reasoning to demonstrate either that it is or is not true on I.

b. $\mathsf{I}[\sim B \to \sim C] \neq \mathsf{T}$

1.	$\mathsf{I}[B] \neq \mathsf{T}$	prem
2.	$\mathsf{I}[\sim B] = \mathsf{T}$	1 ST($\sim$)
3.	$\mathsf{I}[C] = \mathsf{T}$	prem
4.	$\mathsf{I}[\sim C] \neq \mathsf{T}$	3 ST($\sim$)
5.	$\mathsf{I}[\sim B] = \mathsf{T} \vartriangle \mathsf{I}[\sim C] \neq \mathsf{T}$	2,4 cnj
6.	$\mathsf{I}[\sim B \to \sim C] \neq \mathsf{T}$	5 ST($\to$)

It is given that $\mathsf{I}[B] \neq \mathsf{T}$; so by ST($\sim$), $\mathsf{I}[\sim B] = \mathsf{T}$. But it is given that $\mathsf{I}[C] = \mathsf{T}$; so by ST($\sim$), $\mathsf{I}[\sim C] \neq \mathsf{T}$. So $\mathsf{I}[\sim B] = \mathsf{T}$ and $\mathsf{I}[\sim C] \neq \mathsf{T}$; so by ST($\to$), $\mathsf{I}[\sim B \to \sim C] \neq \mathsf{T}$.

E7.2. Produce a formalized derivation, and then informal reasoning to demonstrate each of the following.

a. $A \to B, \sim A \nvDash_s \sim B$

Set $\mathsf{J}[A] \neq \mathsf{T}$, $\mathsf{J}[B] = \mathsf{T}$

1.	$\mathsf{J}[A] \neq \mathsf{T}$	ins (J particular)
2.	$\mathsf{J}[\sim A] = \mathsf{T}$	1 ST($\sim$)
3.	$\mathsf{J}[A] \neq \mathsf{T} \triangledown \mathsf{J}[B] = \mathsf{T}$	1 dsj
4.	$\mathsf{J}[A \to B] = \mathsf{T}$	3 ST($\to$)
5.	$\mathsf{J}[B] = \mathsf{T}$	ins
6.	$\mathsf{J}[\sim B] \neq \mathsf{T}$	5 ST($\sim$)
7.	$\mathsf{J}[A \to B] = \mathsf{T} \vartriangle \mathsf{J}[\sim A] = \mathsf{T} \vartriangle \mathsf{J}[\sim B] \neq \mathsf{T}$	4,2,6 cnj
8.	$S\mathsf{I}(\mathsf{I}[A \to B] = \mathsf{T} \vartriangle \mathsf{I}[\sim A] = \mathsf{T} \vartriangle \mathsf{I}[\sim B] \neq \mathsf{T})$	7 exs
9.	$A \to B, \sim A \nvDash_s \sim B$	8 SV

$\mathsf{J}[A] \neq \mathsf{T}$; so by ST($\sim$), $\mathsf{J}[\sim A] = \mathsf{T}$. But since $\mathsf{J}[A] \neq \mathsf{T}$, $\mathsf{J}[A] \neq \mathsf{T}$ or $\mathsf{J}[B] = \mathsf{T}$; so by ST($\to$), $\mathsf{J}[A \to B] = \mathsf{T}$. And $\mathsf{J}[B] = \mathsf{T}$; so by ST($\sim$), $\mathsf{J}[\sim B] \neq \mathsf{T}$. So $\mathsf{J}[A \to B] = \mathsf{T}$, and $\mathsf{J}[\sim A] = \mathsf{T}$, but $\mathsf{J}[\sim B] \neq \mathsf{T}$; so there is an interpretation I such that $\mathsf{I}[A \to B] = \mathsf{T}$, and $\mathsf{I}[\sim A] = \mathsf{T}$, but $\mathsf{I}[\sim B] \neq \mathsf{T}$; so by SV, $A \to B, \sim A \nvDash_s \sim B$.

b. $A \to B, \sim B \vDash_s \sim A$

1.	$\mid A \to B, \sim B \nvDash_s \sim A$	assp
2.	$\mid S\mathsf{I}(\mathsf{I}[A \to B] = \mathsf{T} \vartriangle \mathsf{I}[\sim B] = \mathsf{T} \vartriangle \mathsf{I}[\sim A] \neq \mathsf{T})$	1 SV
3.	$\mid \mathsf{J}[A \to B] = \mathsf{T} \vartriangle \mathsf{J}[\sim B] = \mathsf{T} \vartriangle \mathsf{J}[\sim A] \neq \mathsf{T}$	2 exs (J particular)
4.	$\mid \mathsf{J}[\sim B] = \mathsf{T}$	3 cnj
5.	$\mid \mathsf{J}[B] \neq \mathsf{T}$	4 ST($\sim$)
6.	$\mid \mathsf{J}[A \to B] = \mathsf{T}$	3 cnj
7.	$\mid \mathsf{J}[A] \neq \mathsf{T} \triangledown \mathsf{J}[B] = \mathsf{T}$	6 ST($\to$)
8.	$\mid \mathsf{J}[A] \neq \mathsf{T}$	7,5 dsj
9.	$\mid \mathsf{J}[\sim A] \neq \mathsf{T}$	3 cnj
10.	$\mid \mathsf{J}[A] = \mathsf{T}$	9 ST($\sim$)
11.	$\mid \perp$	8,10 bot
12.	$A \to B, \sim B \vDash_s \sim A$	1-11 neg

Suppose $A \rightarrow B$, $\sim B \not\models_s \sim A$; then by SV there is an I such that $\mathsf{I}[A \rightarrow B] = \mathsf{T}$ and $\mathsf{I}[\sim B] = \mathsf{T}$ and $\mathsf{I}[\sim A] \neq \mathsf{T}$. Let J be a particular interpretation of this sort; then $\mathsf{J}[A \rightarrow B] = \mathsf{T}$ and $\mathsf{J}[\sim B] = \mathsf{T}$ and $\mathsf{J}[\sim A] \neq \mathsf{T}$. Since $\mathsf{J}[\sim B] = \mathsf{T}$, by ST($\sim$), $\mathsf{J}[B] \neq \mathsf{T}$. And since $\mathsf{J}[A \rightarrow B] = \mathsf{T}$, either $\mathsf{J}[A] \neq \mathsf{T}$ or $\mathsf{J}[B] = \mathsf{T}$; so $\mathsf{J}[A] \neq \mathsf{T}$. But since $\mathsf{J}[\sim A] \neq \mathsf{T}$, by ST($\sim$), $\mathsf{J}[A] = \mathsf{T}$. This is impossible; reject the assumption: $A \rightarrow B$, $\sim B \models_s \sim A$.

E7.4. Complete the demonstration of derived clauses ST′ by completing the demonstration for dst in the other direction (and providing demonstrations for other clauses).

1.	$[(\mathfrak{A} \vartriangle \mathfrak{B}) \triangledown (\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})] \vartriangle \neg[(\neg\mathfrak{A} \triangledown \mathfrak{B}) \vartriangle (\neg\mathfrak{B} \triangledown \mathfrak{A})]$	assp
2.	$(\mathfrak{A} \vartriangle \mathfrak{B}) \triangledown (\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})$	1 cnj
3.	$\neg[(\neg\mathfrak{A} \triangledown \mathfrak{B}) \vartriangle (\neg\mathfrak{B} \triangledown \mathfrak{A})]$	1 cnj
4.	$\neg(\neg\mathfrak{A} \triangledown \mathfrak{B}) \triangledown \neg(\neg\mathfrak{B} \triangledown \mathfrak{A})$	3 dem
5.	$\quad \neg\mathfrak{A} \triangledown \mathfrak{B}$	assp
6.	$\quad \neg(\neg\mathfrak{B} \triangledown \mathfrak{A})$	4,5 dsj
7.	$\quad \mathfrak{B} \vartriangle \neg\mathfrak{A}$	6 dem
8.	$\quad \mathfrak{B}$	7 cnj
9.	$\quad \mathfrak{A} \triangledown \mathfrak{B}$	8 dsj
10.	$\quad \neg(\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})$	9 dem
11.	$\quad \mathfrak{A} \vartriangle \mathfrak{B}$	2,10 dsj
12.	$\quad \mathfrak{A}$	11 cnj
13.	$\quad \neg\mathfrak{A}$	7 cnj
14.	$\quad \bot$	12,13 bot
15.	$\neg(\neg\mathfrak{A} \triangledown \mathfrak{B})$	5-14 neg
16.	$\mathfrak{A} \vartriangle \neg\mathfrak{B}$	15 dem
17.	$\mathfrak{A}$	16 cnj
18.	$\mathfrak{A} \triangledown \mathfrak{B}$	17 dsj
19.	$\neg(\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})$	18 dem
20.	$\mathfrak{A} \vartriangle \mathfrak{B}$	2,19 dsj
21.	$\mathfrak{B}$	20 cnj
22.	$\neg\mathfrak{B}$	16 cnj
23.	$\bot$	21,22 bot
24.	$[(\mathfrak{A} \vartriangle \mathfrak{B}) \triangledown (\neg\mathfrak{A} \vartriangle \neg\mathfrak{B})] \Rightarrow [(\neg\mathfrak{A} \triangledown \mathfrak{B}) \vartriangle (\neg\mathfrak{B} \triangledown \mathfrak{A})]$	1-23 cnd

E7.5. In the non-formalized style, show the following semantic principles for $\leftrightarrow$.

a. *Coms*: $\mathsf{I}[\mathcal{A} \leftrightarrow \mathcal{B}] = \mathsf{T}$ iff $\mathsf{I}[\mathcal{B} \leftrightarrow \mathcal{A}] = \mathsf{T}$.

Suppose $\mathsf{I}[\mathcal{A} \leftrightarrow \mathcal{B}] = \mathsf{T}$; then by ST′($\leftrightarrow$), ($\mathsf{I}[\mathcal{A}] = \mathsf{T}$ and $\mathsf{I}[\mathcal{B}] = \mathsf{T}$) or ($\mathsf{I}[\mathcal{A}] \neq \mathsf{T}$ and $\mathsf{I}[\mathcal{B}] \neq \mathsf{T}$). Suppose $\mathsf{I}[\mathcal{A}] = \mathsf{T}$ and $\mathsf{I}[\mathcal{B}] = \mathsf{T}$; then $\mathsf{I}[\mathcal{B}] = \mathsf{T}$ and $\mathsf{I}[\mathcal{A}] = \mathsf{T}$;

so (I$[\mathcal{B}]$ = T and I$[\mathcal{A}]$ = T) or (I$[\mathcal{B}]$ $\neq$ T and I$[\mathcal{A}]$ $\neq$ T); so by ST$'(\leftrightarrow)$, I$[\mathcal{B} \leftrightarrow \mathcal{A}]$ = T. And similarly if I$[\mathcal{A}]$ $\neq$ T and I$[\mathcal{B}]$ $\neq$ T.

Suppose I$[\mathcal{A} \leftrightarrow \mathcal{B}]$ $\neq$ T; then by ST$'(\leftrightarrow)$, (I$[\mathcal{A}]$ = T and I$[\mathcal{B}]$ $\neq$ T) or (I$[\mathcal{A}]$ $\neq$ T and I$[\mathcal{B}]$ = T). Suppose I$[\mathcal{A}]$ = T and I$[\mathcal{B}]$ $\neq$ T; then I$[\mathcal{B}]$ $\neq$ T and I$[\mathcal{A}]$ = T; so (I$[\mathcal{B}]$ = T and I$[\mathcal{A}]$ $\neq$ T) or (I$[\mathcal{B}]$ $\neq$ T and I$[\mathcal{A}]$ = T); so by ST$'(\leftrightarrow)$, I$[\mathcal{B} \leftrightarrow \mathcal{A}]$ $\neq$ T. And similarly if I$[\mathcal{A}]$ $\neq$ T and I$[\mathcal{B}]$ = T.

So I$[\mathcal{A} \leftrightarrow \mathcal{B}]$ = T iff I$[\mathcal{B} \leftrightarrow \mathcal{A}]$ = T.

E7.6. Using ST($\mid$) as on p. 184, produce non-formalized reasonings to show each of the following.

b. I$[\mathcal{P} \mid (\mathcal{Q} \mid \mathcal{Q})]$ = T iff I$[\mathcal{P} \to \mathcal{Q}]$ = T

By ST($\mid$), I$[\mathcal{P} \mid (\mathcal{Q} \mid \mathcal{Q})]$ = T iff I$[\mathcal{P}]$ $\neq$ T or I$[\mathcal{Q} \mid \mathcal{Q}]$ $\neq$ T; by ST($\mid$), iff I$[\mathcal{P}]$ $\neq$ T or (I$[Q]$ = T and I$[Q]$ = T); iff iff I$[\mathcal{P}]$ $\neq$ T or I$[Q]$ = T; by ST($\to$), iff I$[\mathcal{P} \to \mathcal{Q}]$ = T. So I$[\mathcal{P} \mid (\mathcal{Q} \mid \mathcal{Q})]$ = T iff I$[\mathcal{P} \to \mathcal{Q}]$ = T.

E7.7. Produce non-formalized reasoning to demonstrate each of the following.

b. $\sim(A \leftrightarrow B), \sim A, \sim B \vDash_s C \wedge \sim C$

Suppose $\sim(A \leftrightarrow B), \sim A, \sim B \nvDash_s C \wedge \sim C$; then by SV there is some I such that I$[\sim(A \leftrightarrow B)]$ = T, and I$[\sim A]$ = T, and I$[\sim B]$ = T, but I$[C \wedge \sim C]$ $\neq$ T. Let J be a particular interpretation of this sort; then J$[\sim(A \leftrightarrow B)]$ = T, and J$[\sim A]$ = T, and J$[\sim B]$ = T, but J$[C \wedge \sim C]$ $\neq$ T. From the first, by ST($\sim$), J$[A \leftrightarrow B]$ $\neq$ T; so by ST$'(\leftrightarrow)$, (J$[A]$ = T and J$[B]$ $\neq$ T) or (J$[A]$ $\neq$ T and J$[B]$ = T). But since J$[\sim A]$ = T, by ST($\sim$), J$[A]$ $\neq$ T; so J$[A]$ $\neq$ T or J$[B]$ = T; so it is not the case that J$[A]$ = T and J$[B]$ $\neq$ T; so J$[A]$ $\neq$ T and J$[B]$ = T; so J$[B]$ = T. But J$[\sim B]$ = T; so by ST($\sim$), J$[B]$ $\neq$ T. This is impossible; reject the assumption: $\sim(A \leftrightarrow B), \sim A, \sim B \vDash_s C \wedge \sim C$.

c. $\sim(\sim A \wedge \sim B) \nvDash_s A \wedge B$

Set J$[A]$ = T and J$[B]$ $\neq$ T.

J$[A]$ = T; so by ST($\sim$), J$[\sim A]$ $\neq$ T; so J$[\sim A]$ $\neq$ T or J$[\sim B]$ $\neq$ T; so by ST$'(\wedge)$, J$[\sim A \wedge \sim B]$ $\neq$ T; so by ST($\sim$), J$[\sim(\sim A \wedge \sim B)]$ = T. But it is given that J$[B]$ $\neq$ T; so J$[A]$ $\neq$ T or J$[B]$ $\neq$ T; so by ST$'(\wedge)$, J$[A \wedge B]$ $\neq$ T. So J$[\sim(\sim A \wedge \sim B)]$ = T and J$[A \wedge B]$ $\neq$ T; so by SV, $\sim(\sim A \wedge \sim B) \nvDash_s A \wedge B$.

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Index